MATH 614

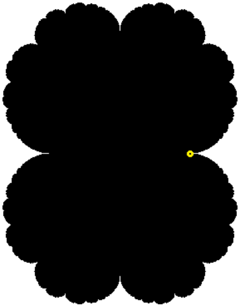
Dynamical Systems and Chaos

Lecture 21: Neutral periodic points. The Julia and Fatou sets.

Neutral fixed points

Example. • $F(z) = z + z^2$.

The map has a fixed point at 0, which is neutral: $F'(0) = 1$. The set D_0 of all points z satisfying $F^n(z) \to 0$ as $n \to \infty$ is open and connected.

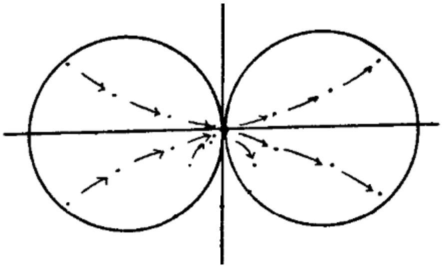


The fixed point 0 is one of the cusp points at the boundary of D_0. The others correspond to eventually fixed points.

Neutral fixed points

Proposition Suppose a function F is holomorphic at 0 and satisfies $F(0) = 0$, $F'(0) = 1$, $F''(0) = 2$ so that $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$.

Then there exists $\mu > 0$ such that **(i)** all points in the disc $D_- = \{z \in \mathbb{C} : |z + \mu| < \mu\}$ are attracted to 0; and **(ii)** all points in the disc $D_+ = \{z \in \mathbb{C} : |z - \mu| < \mu\}$ are repelled from 0.



Proof: We change coordinates using the function $H(z) = 1/z$, which maps the discs D_- and D_+ onto halfplanes $\mathrm{Re}\, z < -1/(2\mu)$ and $\mathrm{Re}\, z > 1/(2\mu)$.

The function F is changed to $G(z) = 1/F(1/z)$. Since $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$, it follows that

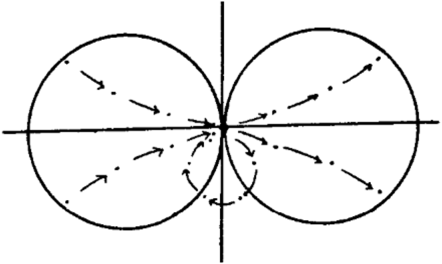
$$\begin{aligned} F(1/z) &= z^{-1} + z^{-2} + O(|z|^{-3}) \\ &= z^{-1}\big(1 + z^{-1} + O(|z|^{-2})\big) \quad \text{as} \quad z \to \infty. \end{aligned}$$

Then

$$\begin{aligned} G(z) &= z\big(1 + z^{-1} + O(|z|^{-2})\big)^{-1} \\ &= z\big(1 - z^{-1} + O(|z|^{-2})\big) = z - 1 + O(|z|^{-1}). \end{aligned}$$

If μ is small enough, then the halfplane $\mathrm{Re}\, z < -1/(2\mu)$ is invariant under the map G while the halfplane $\mathrm{Re}\, z > 1/(2\mu)$ is invariant under G^{-1}.

The proposition suggests that for most of the points in a neighborhood of 0, the forward and backward orbits under the map F both converge to 0.



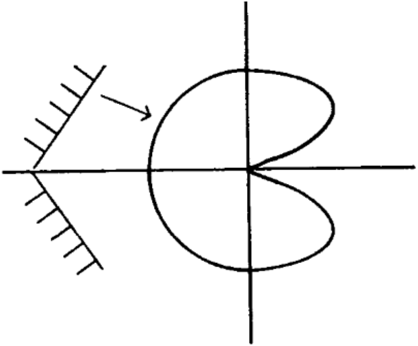
Examples. • $F(z) = \dfrac{z}{1-z}$.

This is a Möbius transformation with 0 the only fixed point. It follows that all forward and backward orbits converge to 0.

• $F(z) = z + z^2$.

The orbits of all points on the ray $z > 0$ converge to ∞ and so are the orbits of all points in a small cusp about this ray.

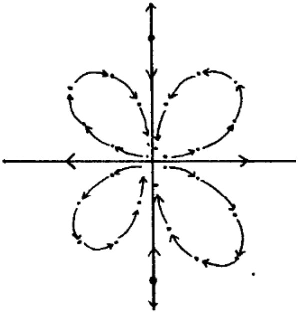
In the proof of the proposition, we could use wedge-shaped regions instead of halfplanes. This would allow to extend basins of attraction from discs to cardioid-shaped regions.



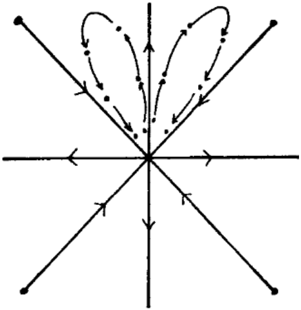
In the case not all points near 0 are attracted to 0, the set of points that are attracted is locally a simply connected domain with 0 on its boundary. This domain is called the **attracting petal** of the fixed point 0. Similarly, there is also the **repelling petal** of 0.

More types of neutral fixed points

$F_1(z) = z + z^3$



$F_2(z) = z + z^5$



In the first example, there are two attracting and two repelling petals of the fixed point 0. In the second example, there are 4 attracting and 4 repelling petals.

Siegel discs

Theorem (Siegel) Let F be a holomorphic function at z_0 such that $F(z_0) = z_0$ and $F'(z_0) = e^{2\pi i\alpha}$, where α is irrational. Suppose that α is not very well approximated by rational numbers, namely, $|\alpha - p/q| > aq^{-b}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}$. Then there is a neighborhood U of z_0 on which the function F is analytically conjugate to the irrational rotation $L(z) = e^{2\pi i\alpha}z$.

The domain U is called a **Siegel disc**.

In the case α is well approximated by rational numbers, it can happen that the fixed point z_0 is a limit point of other periodic points of the map F.

The Julia set

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Informally, the Julia set of P is the set of points where iterates of P exhibit sensitive dependence on initial conditions (chaotic behavior). The Fatou set of P is the set of points where iterates of P exhibit regular, stable behavior.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P.

Example. $Q_0 : \mathbb{C} \to \mathbb{C}$, $Q_0(z) = z^2$.

0 is an attracting fixed point. The other periodic points are located on the unit circle $|z| = 1$. All of them are repelling.

Any point of the form

$$\exp\Bigl(2\pi i \frac{m}{n}\Bigr),$$

where m, n are integers and n is odd, is periodic. Hence periodic points are dense in the unit circle.

Thus $J(Q_0) = \{z \in \mathbb{C} : |z| = 1\}$.

Quadratic family

$Q_c : \mathbb{C} \to \mathbb{C},\ c \in \mathbb{C}.$
$Q_c(z) = z^2 + c.$

Theorem $J(Q_{-2}) = [-2, 2].$

Proof: The map $H(z) = z + z^{-1}$ is holomorphic on $R = \{z \in \mathbb{C} : |z| > 1\}$. It maps R onto $\mathbb{C} \setminus [-2, 2]$ in a one-to-one way (a conformal map). Also, H maps each of the semicircles

$$\{e^{i\phi} \mid 0 \leq \phi \leq \pi\} \quad \text{and} \quad \{e^{i\phi} \mid -\pi \leq \phi \leq 0\}$$

homeomorphically onto $[-2, 2]$.
Finally, $H(Q_0(z)) = Q_{-2}(H(z))$ for all $z \neq 0$.

Normal family

Let $\mathcal{F}$ be a collection of holomorphic functions defined in a domain $U \subset \mathbb{C}$.

Definition. The collection $\mathcal{F}$ is a **normal family** in U if every sequence $F_1, F_2, \ldots$ of functions from $\mathcal{F}$ has a subsequence $F_{n_1}, F_{n_2}, \ldots$ $(1 \le n_1 < n_2 < \ldots)$ which either

(i) converges uniformly on compact subsets of U, or

(ii) converges uniformly to ∞ on U.

The condition (i) means that there exists a function $f : U \to \mathbb{C}$ such that for any compact set $D \subset U$ we have

$$\sup_{z \in D} |F_{n_k}(z) - f(z)| \to 0 \text{ as } k \to \infty.$$

The function f is going to be continuous.

The condition (ii) means that for any $R > 0$ there exists an integer $K > 0$ such that $|F_{n_k}(z)| > R$ for all $k \ge K$ and $z \in U$.

The Fatou set

Let $\mathcal{F}$ be a collection of holomorphic functions defined in a domain $U \subset \overline{\mathbb{C}}$.

We say that the collection $\mathcal{F}$ is **normal at** a finite point $z \in U$ if it is a normal family in some neighborhood of z. In the case $\infty \in U$, we say $\mathcal{F}$ is **normal at infinity** if the collection of functions $G(z) = F(1/z)$, $F \in \mathcal{F}$ is normal at 0.

Definition. The **Fatou set** $S(P)$ of a holomorphic map $P : U \to U$ is the set of all points $z \in U$ such that the family of iterates $\{P^n\}_{n \geq 1}$ is normal at z.

By definition, the Fatou set is open.