MATH 614

Dynamical Systems and Chaos

Lecture 25:
Ergodic theorems.
Ergodicity and mixing.

Measure-preserving transformation

Definition. A **measured space** is a triple $(X, \mathcal{B}, \mu)$, where X is a set, $\mathcal{B}$ is a σ-algebra of (measurable) subsets of X, and $\mu : \mathcal{B} \to [0, \infty]$ is a σ-additive measure on X (finite or σ-finite).

A mapping $T : X \to X$ is called **measurable** if preimage of any measurable set under T is also measurable: $E \in \mathcal{B} \implies T^{-1}(E) \in \mathcal{B}$.

A measurable mapping $T : X \to X$ is called **measure-preserving** if for any $E \in \mathcal{B}$ one has $\mu(T^{-1}(E)) = \mu(E)$.

Borel sets

Proposition Given a collection S of subsets of X, there exists a minimal σ-algebra of subsets of X that contains S.

Suppose X is a topological space. The **Borel σ-algebra** $\mathcal{B}(X)$ is the minimal σ-algebra that contains all open subsets of X. Elements of $\mathcal{B}(X)$ are called **Borel sets**.

A mapping $F : X \to X$ is measurable relative to $\mathcal{B}(X)$ if and only if the preimage of any open set is Borel. In particular, each continuous map is measurable.

Recurrence

$(X, \mathcal{B}, \mu)$: measured space
$T : X \to X$: measure-preserving mapping

Let E be a measurable subset of X. A point $x \in E$ is called **recurrent** if $T^n(x) \in E$ for some $n \geq 1$.

A point $x \in E$ is called **infinitely recurrent** if the orbit $x, T(x), T^2(x), \ldots$ visits E infinitely many times.

Theorem (Poincaré) Suppose μ is a finite measure. Then almost all points of E are infinitely recurrent.

Individual ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

Birkhoff's Ergodic Theorem For any function $f \in L_1(X, \mu)$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f^*(x)$$

exists for almost all $x \in X$. The function f^* is T-invariant, i.e., $f^* \circ T = f^*$ almost everywhere. If μ is finite then $f^* \in L_1(X, \mu)$ and

$$\int_X f^* \, d\mu = \int_X f \, d\mu.$$

Ergodicity

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

We say that a measurable set $E \subset X$ is **invariant** under T if $\mu(E \triangle T^{-1}(E)) = 0$, that is, if $E = T^{-1}(E)$ up to a set of zero measure.

Note that there is a measurable set $E_0 \subset E$ such that $\mu(E \triangle E_0) = 0$ and $T^{-1}(E_0) = E_0$. Namely, let $E_1 = E \cup T^{-1}(E) \cup T^{-2}(E) \cup \dots$. Then $E \subset E_1$, $\mu(E_1 \setminus E) = 0$, $\mu(E_1 \triangle T^{-1}(E_1)) = 0$, and $T^{-1}(E_1) \subset E_1$. Now $E_0 = E_1 \cap T^{-1}(E_1) \cap T^{-2}(E_1) \cap \dots$.

Definition. The transformation T is called **ergodic** with respect to μ if any T-invariant measurable set E has either zero or full measure: $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Birkhoff's Ergodic Theorem (ergodic case)

Suppose μ is finite and T is ergodic. Given $f \in L_1(X, \mu)$, for almost all $x \in X$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \frac{1}{\mu(X)} \int_X f\, d\mu.$$

(time average is equal to space average)

In the case $f = \chi_E$ $(E \in \mathcal{B})$, we obtain

$$\lim_{n\to\infty} \frac{\#\{0 \le k \le n-1 \mid T^k(x) \in E\}}{n} = \frac{\mu(E)}{\mu(X)}.$$

(almost every orbit is uniformly distributed)

Koopman's operator

$(X, \mathcal{B}, \mu)$: measured space
$T : X \to X$: measure-preserving transformation

To any function $f : X \to \mathbb{C}$ we assign another function Uf defined by $(Uf)(x) = f(T(x))$ for all $x \in X$.

Linear functional operator U: $f \mapsto Uf$.

Proposition If f is integrable then so is Uf. Moreover,

$$\int_X Uf\, d\mu = \int_X f(T(x))\, d\mu(x) = \int_X f\, d\mu.$$

$f \in L_2(X, \mu)$ means that $\int_X |f|^2 \, d\mu < \infty$.

$L_2(X, \mu)$ is a Hilbert space with respect to the inner product

$$(f, g) = \int_X f(x)\overline{g(x)} \, d\mu(x).$$

Let T be a measure-preserving transformation and U be the associated operator.

Then $U(L_2(x, \mu)) \subset L_2(X, \mu)$. Furthermore,

$$(Uf, Ug) = (f, g)$$

for all $f, g \in L_2(X, \mu)$.

That is, U is an **isometric** operator in the Hilbert space $L_2(X, \mu)$. If T is invertible and T^{-1} is also measure-preserving, then U is a **unitary** operator.

Mean ergodic theorem

von Neumann's Ergodic Theorem Suppose U is an isometric operator in a Hilbert space $\mathcal{H}$. Then for any $f \in \mathcal{H}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f = f^* \text{ (in } \mathcal{H}\text{)},$$

where $f^* \in \mathcal{H}$ is the orthogonal projection of f on the subspace of U-invariant functions in $\mathcal{H}$.

Namely, $Uf^* = f^*$ and $(f - f^*, g) = 0$ for any element $g \in \mathcal{H}$ such that $Ug = g$.

If U is associated to a measure-preserving map $T : X \to X$, then for any $f \in L_2(X, \mu)$ we have

$$\lim_{n\to\infty} \int_X \Big| \frac{1}{n} \sum\nolimits_{k=0}^{n-1} U^k f - f^* \Big|^2 d\mu \to 0,$$

where $f^* \in L_2(X, \mu)$ and $Uf^* = f^*$.

Lemma T is ergodic if and only if $Uf = f$ for a measurable function f implies f is constant (almost everywhere).

If T is ergodic then

$$\lim_{n\to\infty} \int_X \Big| \frac{1}{n} \sum\nolimits_{k=0}^{n-1} U^k f - c \Big|^2 d\mu \to 0,$$

where

$$c = \frac{1}{\mu(X)} \int_X f \, d\mu.$$

Rotations of the circle

Measured space $(S^1, \mathcal{B}(S^1), \mu)$, where μ is the length measure on S^1.

R_α: rotation of the unit circle by angle α.
R_α is a measure-preserving homeomorphism.

Theorem If α is not commensurable with π, then the rotation R_α is ergodic.

Let U_α be the associated operator on $L_2(S^1, \mu)$.

Relative to the angular coordinate on S^1, elements of $L_2(S^1, \mu)$ are 2π-periodic functions on $\mathbb{R}$. The inner product is given by

$$(f, g) = \int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

The operator U_α acts as follows:

$$(U_\alpha f)(x) = f(x + \alpha), \;\; x \in \mathbb{R}.$$

For any $m \in \mathbb{Z}$ let $h_m(x) = e^{imx}$, $x \in \mathbb{R}$. Then $h_m \in L_2(S^1, \mu)$ and $U_\alpha h_m = e^{im\alpha} h_m$ so that h_m is an eigenfunction of U_α. Note that $\{h_m\}_{m\in\mathbb{Z}}$ is an orthogonal basis of the Hilbert space $L_2(X, \mu)$. We say that U_α has **pure point spectrum**.

Any $f \in L_2(X, \mu)$ is uniquely expanded as

$$f = \sum_{m\in\mathbb{Z}} c_m h_m, \quad \text{(Fourier series)}$$

where $c_m \in \mathbb{C}$. Then

$$U_\alpha f = \sum_{m\in\mathbb{Z}} e^{im\alpha} c_m h_m.$$

Hence $U_\alpha f = f$ only if $(e^{im\alpha} - 1)c_m = 0$ for all $m \in \mathbb{Z}$. That is, if $f = c_0$, a constant.

Mixing

$(X, \mathcal{B}, \mu)$: measured space of finite measure
$T : X \to X$: measure-preserving transformation

T is called **mixing** if for any measurable sets $A, B \subset X$ we have

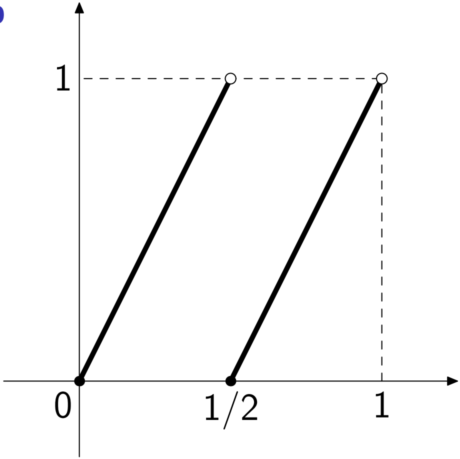
$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \frac{\mu(A)\mu(B)}{\mu(X)}.$$

Lemma Mixing $\implies$ ergodicity.

Proof: Suppose $C \subset X$ is measurable and T-invariant. Then $T^{-n}(C) = C$ up to a set of zero measure. Therefore $\mu(T^{-n}(C) \cap C) = \mu(C)$.

If T is mixing then $\mu(C) = \mu(C)\mu(C)/\mu(X)$, which implies that $\mu(C) = 0$ or $\mu(C) = \mu(X)$.

Doubling map



$D : [0, 1) \to [0, 1)$,
$D(x) = 2x \bmod 1, \;\; x \in [0, 1)$.

Theorem The doubling map is mixing.

Proof: Let $A \subset [0,1)$ and $n \geq 1$. Then $D^{-n}(A)$ is the union of 2^n disjoint pieces $\frac{1}{2^n}A + \frac{k}{2^n}$, $k = 0, 1, \ldots, 2^n - 1$.

Suppose $B = [\frac{l}{2^m}, \frac{l+1}{2^m})$, where $m > 0$, $0 \leq l < 2^m$. If $n \geq m$ then exactly 2^{n-m} pieces are contained in B, the others are disjoint from B. Hence

$$\mu(D^{-n}(A) \cap B) = 2^{n-m} \cdot 2^{-n} \mu(A) = \mu(A)\mu(B).$$

Since any measurable set B can be approximated by disjoint unions of the above intervals,

$$\lim_{n \to \infty} \mu(D^{-n}(A) \cap B) = \mu(A)\mu(B).$$

Proposition The rotation R_α of the circle is not mixing.

Proof: For any $\varepsilon > 0$ there exists $n > 0$ such that $R_\alpha^n = R_{n\alpha}$ is the rotation by an angle $< \varepsilon$.

Hence there exists a sequence $n_1 < n_2 < \dots$ such that for any arc $\gamma \subset S^1$,

$$\lim_{k\to\infty} \mu(R_\alpha^{-n_k}(\gamma) \cap \gamma) = \mu(\gamma).$$

But $\mu(\gamma) \neq \mu(\gamma)\mu(\gamma)/\mu(S^1)$ if $\gamma \neq S^1$.

$(X, \mathcal{B}, \mu)$: measured space of finite measure
$T : X \to X$: measure-preserving transformation

T is mixing if and only if for any $f, g \in L_2(X, \mu)$,

$$\lim_{n\to\infty} \int_X f(T^n(x))g(x)\, d\mu(x) = \frac{1}{\mu(X)} \int_X f\, d\mu \int_X g\, d\mu.$$

$$\lim_{n\to\infty} (U^n f, g) = \frac{(f,1)(1,g)}{(1,1)}.$$

Suppose f is a nonconstant eigenfunction of U, $Uf = \lambda f$, $|\lambda| = 1$. It is no loss to assume that $(f, 1) = 0$. Obviously,

$$(U^n f, f) = \lambda^n (f, f) \not\to 0 \ \text{ as } \ n \to \infty.$$

$(X, \mathcal{B}, \mu)$: measured space of finite measure
$T : X \to X$: measure-preserving transformation
$U : L_2(X, \mu) \to L_2(X, \mu)$: associated linear operator

T is called **weakly mixing** if U has no eigenfunctions other than constants.

$$\text{mixing} \implies \text{weak mixing} \implies \text{ergodicity}$$

In particular, the doubling map has no nonconstant eigenfunctions. In this case, the operator U has **countable Lebesgue spectrum**. Namely, there are functions f_{nm} $(n, m = 1, 2, \dots)$ on S^1 such that 1 and f_{nm}, $n, m \geq 1$ form an orthogonal basis for $L_2(X, \mu)$, and $Uf_{nm} = f_{n,m+1}$ for any $n, m \geq 1$.

Translation of the torus $R_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$, $\alpha, \beta \in \mathbb{R}$.
$R_{\alpha,\beta}(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$.
$R_{\alpha,\beta}$ is a measure-preserving homeomorphism.

Theorem $R_{\alpha,\beta}$ has pure point spectrum. It is ergodic if and only if the numbers α, β, and 1 are linearly independent over $\mathbb{Q}$ (i.e., for any $k, m, n \in \mathbb{Z}$ the equality $k\alpha + m\beta + n = 0$ implies $k = m = n = 0$).

Theorem Any hyperbolic toral automorphism T_A of the flat torus is ergodic.

Proof: Let $f : \mathbb{T}^2 \to \mathbb{C}$ be a continuous function. By Birkhoff's Ergodic Theorem, for almost all $x \in \mathbb{T}^2$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_A^k(x)) = f^*(x),$$

where f^* is an integrable function. Also, for almost all $x \in \mathbb{T}^2$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_A^{-k}(x)) = f^{**}(x),$$

where f^{**} is an integrable function.

von Neumann's Ergodic Theorem implies that
$f^* = f^{**}$ almost everywhere.

Let $x \in \mathbb{T}^2$ and $y \in W^s(x)$. Then
$\mathrm{dist}(T_A^n(y), T_A^n(x)) \to 0$ as $n \to \infty$.
Since f is continuous, it follows that
$|f(T_A^n(y)) - f(T_A^n(x))| \to 0$ as $n \to \infty$.
Therefore $f^*(y) = f^*(x)$.
Similarly, if $y \in W^u(x)$ then $f^{**}(y) = f^{**}(x)$.

Thus f^* is constant along leaves of the stable foliation while f^{**} is constant along leaves of the unstable foliation. Since $f^* = f^{**}$ a.e., it follows that f^* is constant almost everywhere.

Doubling map $D_2 : \mathbb{T}^2 \to \mathbb{T}^2$;
$D_2(x_1, x_2) = (2x_1 \bmod 1, 2x_2 \bmod 1)$.

Theorem The doubling map on the torus preserves measure and is mixing. It has countable Lebesgue spectrum.