MATH 614

Dynamical Systems and Chaos

Lecture 8:
Topological conjugacy.

Topological conjugacy

Suppose $f : X \to X$ and $g : Y \to Y$ are transformations of topological spaces.

Definition. We say that a map $\phi : X \to Y$ is a **semi-conjugacy** of f with g if ϕ is onto and $\phi \circ f = g \circ \phi$.

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} & X \\ \phi \Big\downarrow & & \Big\downarrow \phi \\ Y & \stackrel{g}{\longrightarrow} & Y \end{array}$$

The map ϕ is a **conjugacy** if, additionally, it is invertible. The map ϕ is a **topological conjugacy** if, additionally, it is a homeomorphism, which means that both ϕ and ϕ^{-1} are continuous. In the latter case, we say that the maps f and g are **topologically conjugate**. Note that $f = \phi^{-1} g \phi$ and $g = \phi f \phi^{-1}$.

Suppose $f : X \to X$ and $g : Y \to Y$ are transformations of topological spaces and $\phi : X \to Y$ is a semi-conjugacy of f with g.

- ϕ maps any orbit of f onto an orbit of g (both as a sequence and a set). Indeed, $\phi \circ f = g \circ \phi$ implies that $\phi \circ f^n = g^n \circ \phi$ for all $n \geq 1$.

- If x is a periodic point of f, then $\phi(x)$ is a periodic point of g. In the case ϕ is invertible, the prime period of $\phi(x)$ is the same as that of x.

- If x is an eventually periodic point of f, then $\phi(x)$ is an eventually periodic point of g.

- In the case ϕ is a topological conjugacy, if x is a weakly attracting periodic point of f, then $\phi(x)$ is a weakly attracting periodic point of g. Similarly, if x is a weakly repelling periodic point of f, then $\phi(x)$ is a weakly repelling periodic point of g.

Examples of topological conjugacy

- Linear maps $f(x) = \lambda x$ and $g(x) = \mu x$ on $\mathbb{R}$ are topologically conjugate if $0 < \lambda, \mu < 1$ or if $\lambda, \mu > 1$. If $0 < \lambda < 1 < \mu$, then they are not topologically conjugate.

- The maps $f(x) = x/2$, $g(x) = x^3$, and $h(x) = x - x^3$ are topologically conjugate on $[-1/2, 1/2]$. (For each map 0 is a fixed point and all orbits converge to 0. However the fixed point is attracting for f, super-attracting for g, and only weakly attracting for h.)

- Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map and Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0, 1]$. If the itinerary map $S : \Lambda \to \Sigma_{\{0,1\}}$ is one-to-one, then it provides topological conjugacy of the restriction $f|_\Lambda$ of the map f to Λ with the shift $\sigma : \Sigma_{\{0,1\}} \to \Sigma_{\{0,1\}}$. In general, S is a continuous semi-conjugacy.

Topological conjugacy of linear maps

Consider the family of linear maps $f_\lambda : \mathbb{R} \to \mathbb{R}$ given by $f_\lambda(x) = \lambda x,\ x \in \mathbb{R},$ where λ is a real parameter.

Let us also define another family of maps $\phi_\alpha : \mathbb{R} \to \mathbb{R}$ depending on a parameter $\alpha > 0$:

$$\phi_\alpha(x) = \begin{cases} x^\alpha & \text{if } x \geq 0, \\ -|x|^\alpha & \text{if } x < 0. \end{cases}$$

Note that ϕ_α is a homeomorphism and $\phi_\alpha^{-1} = \phi_{1/\alpha}$. For any $\lambda, x \geq 0$,

$$\phi_\alpha f_\lambda \phi_\alpha^{-1}(x) = \phi_\alpha f_\lambda(x^{1/\alpha}) = \phi_\alpha(\lambda x^{1/\alpha}) = (\lambda x^{1/\alpha})^\alpha = \lambda^\alpha x.$$

Since $f_\lambda(-x) = -f_\lambda(x)$ and $\phi_\alpha(-x) = -\phi_\alpha(x)$ for all x, the same equality holds for $\lambda \geq 0$ and $x < 0$. Similarly, for $\lambda < 0$ and any $x \in \mathbb{R}$ we obtain $\phi_\alpha f_\lambda \phi_\alpha^{-1}(x) = -|\lambda|^\alpha x$.

Therefore $\phi_\alpha f_\lambda \phi_\alpha^{-1} = f_{\lambda'}$, where $\lambda' = \phi_\alpha(\lambda)$.

Proposition Two linear maps f_λ and $f_{\lambda'}$ are topologically conjugate if and only if one of the following conditions holds: **(i)** $\lambda, \lambda' < -1$, **(ii)** $\lambda = \lambda' = -1$, **(iii)** $-1 < \lambda, \lambda' < 0$, **(iv)** $\lambda = \lambda' = 0$, **(v)** $0 < \lambda, \lambda' < 1$, **(vi)** $\lambda = \lambda' = 1$, **(vii)** $\lambda, \lambda' > 1$.

Proof: If one of the seven conditions holds, then $\lambda' = \phi_\alpha(\lambda)$ for some $\alpha > 0$. It follows that $\phi_\alpha f_\lambda \phi_\alpha^{-1} = f_{\lambda'}$, in particular, f_λ and $f_{\lambda'}$ are topologically conjugate.

If neither condition holds, we need to distinguish f_λ from $f_{\lambda'}$ by a property invariant under topological conjugacy. First notice that f_0 is the only linear map that is not one-to-one. Further, f_1 is the identity map and f_{-1} is distinguished since f_{-1}^2 is the identity map while f_{-1} is not. The only fixed point 0 of f_λ is attracting if $|\lambda| < 1$ and repelling if $|\lambda| > 1$. Finally, for any $x \neq 0$ the interval with endpoints x and $f_\lambda(x)$ contains the fixed point 0 if $\lambda < 0$ and does not if $\lambda > 0$.

Proposition 1 Suppose $f : [0, a] \to \mathbb{R}$ and $g : [0, b] \to \mathbb{R}$ are continuous maps such that $f(0) = g(0) = 0$, $f(x) < x$ for $0 < x \le a$, and $g(x) < x$ for $0 < x \le b$. Then f and g are topologically conjugate.

Let $U = (f(a), a)$. Then U is a **wandering domain** of the map f, which means that sets $U, f(U), f^2(U), \dots$ are disjoint. Similarly, $V = (g(b), b)$ is a wandering domain of g.

$$\begin{array}{ccccccc} U & \xrightarrow{f} & f(U) & \xrightarrow{f} & f^2(U) & \xrightarrow{f} & \dots \\ \phi\big\downarrow & & & & & & \\ V & \xrightarrow{g} & g(V) & \xrightarrow{g} & g^2(V) & \xrightarrow{g} & \dots \end{array}$$

Proposition 2 Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable maps such that $f(0) = g(0) = 0$, $0 < f'(x) < 1$ and $0 < g'(x) < 1$ for all $x \in \mathbb{R}$. Then f and g are topologically conjugate.