

MATH 614

Dynamical Systems and Chaos

Lecture 12:
Sharkovskii's theorem (continued).

Sharkovskii's Theorem

The **Sharkovskii ordering** is the following strict linear ordering of the natural numbers:

$$\begin{array}{ccccccc} & 3 & \triangleright & 5 & \triangleright & 7 & \triangleright & 9 & \triangleright & \dots \\ \triangleright & 2 \cdot 3 & \triangleright & 2 \cdot 5 & \triangleright & 2 \cdot 7 & \triangleright & 2 \cdot 9 & \triangleright & \dots \\ \triangleright & 2^2 \cdot 3 & \triangleright & 2^2 \cdot 5 & \triangleright & 2^2 \cdot 7 & \triangleright & 2^2 \cdot 9 & \triangleright & \dots \\ & \dots & & \dots & & \dots & & \dots & & \dots \\ \dots & \triangleright & 2^k & \triangleright & \dots & \triangleright & 2^3 & \triangleright & 2^2 & \triangleright & 2 & \triangleright & 1. \end{array}$$

Theorem 1 Suppose $f : J \rightarrow J$ is a continuous map of an interval $J \subset \mathbb{R}$. If f admits a periodic point of prime period n and $n \triangleright m$ for some $m \in \mathbb{N}$, then f admits a periodic point of prime period m as well.

Theorem 2 Suppose P is a set of natural numbers such that $n \in P$ and $n \triangleright m$ imply $m \in P$ for all $m, n \in \mathbb{N}$. Then there exists a continuous map $f : \mathbb{R} \rightarrow \mathbb{R}$ with P as the set of prime periods of its periodic points.

Suppose $f : J \rightarrow J$ is a continuous map of an interval $J \subset \mathbb{R}$. Given two closed bounded intervals $I_1, I_2 \subset J$, we write and draw $I_1 \rightarrow I_2$ if $f(I_1) \supset I_2$ (i.e., if I_1 covers I_2 under the action of f).

Lemma 1 If $I \rightarrow I$, then the interval I contains a fixed point of the map f .

Lemma 2 If the map f has a periodic orbit, then it has a fixed point.

Lemma 3 If $I \rightarrow I'$, then there exists a closed interval $I_0 \subset I$ such that f maps I_0 onto I' .

Lemma 4 If $I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_n \rightarrow I_1$, then there exists a fixed point x of f^n such that $x \in I_1$, $f(x) \in I_2, \dots, f^{n-1}(x) \in I_n$.

Proposition 5 If the map f has a periodic point of prime period 3, then it has periodic points of any prime period.

Proof: Suppose x_1, x_2, x_3 are points forming a periodic orbit of f , ordered so that $x_1 < x_2 < x_3$. We have that either $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$, or else $f(x_1) = x_3$, $f(x_2) = x_1$, $f(x_3) = x_2$. In the first case, let $I_1 = [x_2, x_3]$ and $I_2 = [x_1, x_2]$. Otherwise we let $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. Then $I_1 \xrightarrow{\circlearrowleft} I_2 \xrightarrow{\circlearrowleft} I_1$, i.e., $I_1 \rightarrow I_2 \rightarrow I_1$ and $I_2 \rightarrow I_1$.

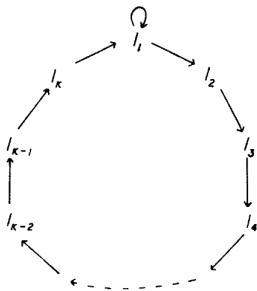
The map f has a periodic point of prime period 3. By Lemma 2, it also has a fixed point. To find a periodic point of prime period n , where $n = 2$ or $n \geq 4$, we notice that

$$I_2 \rightarrow \underbrace{I_1 \rightarrow I_1 \rightarrow \cdots \rightarrow I_1}_{n-1 \text{ times}} \rightarrow I_2.$$

By Lemma 4, there exists $x \in I_2$ such that $f^n(x) = x$ and $f^i(x) \in I_1$ for $1 \leq i \leq n-1$. If $x \notin I_1$, we obtain that n is the prime period of x . Otherwise $x = x_2$, which leads to a contradiction.

Proposition 6 If the map f has a periodic point of odd prime period $n \geq 5$, then it has a periodic point of any prime period $m \triangleleft n$.

Proof: It is no loss to assume that f has no periodic points of odd prime periods p , $1 < p < n$. Let x_1, x_2, \dots, x_n be points of a periodic orbit of prime period n , $x_1 < x_2 < \dots < x_n$. First we show that one can choose $k \geq 2$ distinct intervals I_1, I_2, \dots, I_k among $[x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ that satisfy



Then we show that, in fact, $k = n - 1$.

First we show that one can choose $k \geq 2$ distinct intervals I_1, I_2, \dots, I_k among $[x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ that satisfy $I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_k \rightarrow I_1$ and $I_1 \rightarrow I_1$.

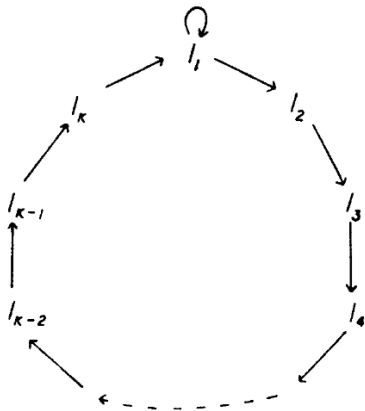
Let $I_1 = [x_j, x_{j+1}]$, where j is the largest index satisfying $f(x_j) > x_j$. Then $f(x_j) \geq x_{j+1}$ and $f(x_{j+1}) \leq x_j$, which implies that $I_1 \rightarrow I_1$.

Further, there is an interval $I_\infty = [x_i, x_{i+1}] \neq I_1$ such that $f(x_i)$ and $f(x_{i+1})$ are on different sides of I_1 so that $I_\infty \rightarrow I_1$. Indeed, otherwise f would move each x_i to the other side of I_1 , which is impossible since n is odd.

Next there are intervals I_2, \dots, I_k of the form $[x_\ell, x_{\ell+1}]$ such that I_1, I_2, \dots, I_k are distinct and $I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_k = I_\infty$.

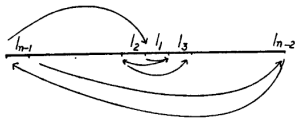
Clearly, $k \leq n - 1$. In fact, $k = n - 1$ as otherwise we would get a periodic orbit of prime period $n - 2$ from the chain

$$I_k \rightarrow \underbrace{I_1 \rightarrow I_1 \rightarrow \dots \rightarrow I_1}_{n-k-1 \text{ times}} \rightarrow I_2 \rightarrow I_3 \rightarrow \dots \rightarrow I_k.$$

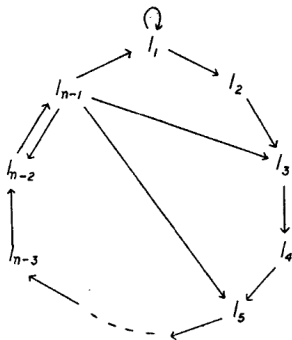


For any diagram of this kind, $k = n - 1$.

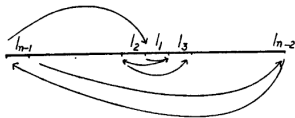
As a consequence, $I_s \not\rightarrow I_t$ if $t > s + 1$ and $I_s \not\rightarrow I_1$ if $1 < s < n - 1$. It follows that, up to the mirror image, there is only one possible ordering of the intervals I_1, I_2, \dots, I_{n-1} :



This leads to a more refined diagram of coverings:



As a consequence, $I_s \not\rightarrow I_t$ if $t > s + 1$ and $I_s \not\rightarrow I_1$ if $1 < s < n - 1$. It follows that, up to the mirror image, there is only one possible ordering of the intervals I_1, I_2, \dots, I_{n-1} :



This leads to a more refined diagram of coverings:

$I_1 \rightarrow I_1$, $I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_{n-1} \rightarrow I_1$, and $I_{n-1} \rightarrow I_{n-2s}$.

We use this diagram and Lemma 4 to obtain a periodic orbit of f of prime period m for every natural number $m \triangleleft n$.

Namely, in the case $m \geq n - 1$ we use a chain

$$I_{n-1} \rightarrow \underbrace{I_1 \rightarrow I_1 \rightarrow \dots \rightarrow I_1}_{m-n+2 \text{ times}} \rightarrow I_2 \rightarrow I_3 \rightarrow \dots \rightarrow I_{n-1}.$$

In the case $1 < m < n - 1$, the number m is even, $m = 2s$, and we use a chain $I_{n-1} \rightarrow I_{n-2s} \rightarrow I_{n-2s+1} \rightarrow \dots \rightarrow I_{n-1}$.

Finally, in the case $m = 1$, we use the chain $I_1 \rightarrow I_1$.

Lemma 7 $2n \triangleright 2m$ if and only if $n \triangleright m$ for all $n, m \in \mathbb{N}$.

Lemma 8 If x is a periodic point of the map f of prime period n , then x is also a periodic point of f^k of prime period $n/\gcd(n, k)$.

Lemma 9 Assume that for some $n, m > 1$, period n implies period m . Then period $2n$ implies period $2m$.

Proof: Suppose x is a periodic point of the map f of prime period $2n$. Then x is a periodic point of f^2 of prime period n . By assumption, f^2 also has a periodic point y of prime period m . Then $f^{2m}(y) = (f^2)^m(y) = y$ so that y is a periodic point of f of prime period ℓ , where ℓ divides $2m$. By Lemma 8, $\ell = 2m$ if ℓ is even and $\ell = m$ if ℓ is odd. In the former case, we are done. In the latter case, we apply Proposition 5 or 6.

Lemma 10 If f has a periodic point of prime period 4, then it also has a periodic point of prime period 2.

On the converse of Sharkovskii's Theorem

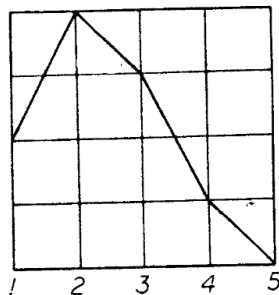
Let $n \in \mathbb{N}$. Consider an arbitrary permutation π of $\{1, 2, \dots, n\}$ that consists of a single cycle of length n .

We can extend π to a continuous function $f : [1, n] \rightarrow [1, n]$ so that f be linear on each of the intervals $[1, 2], [2, 3], \dots, [n-1, n]$. Further, we can extend f to a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ so that f be constant on $(-\infty, 1]$ and on $[n, \infty)$. Then all periodic points of f are in $[1, n]$.

By construction, f has a periodic point of prime period n . One can try to pick π so that there are no periodic points of prime periods $m \triangleright n$.

Period 5 orbit, but no period 3 orbit

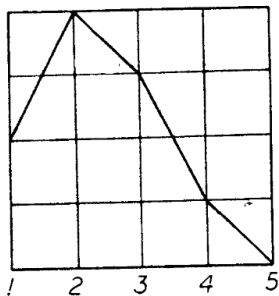
Example. $n = 5$, $\pi = (13425)$.



We obtain that $f^3([1, 2]) = [2, 5]$, $f^3([2, 3]) = [3, 5]$,
 $f^3([3, 4]) = [1, 5]$, $f^3([4, 5]) = [1, 4]$. Moreover, f^3 is strictly
decreasing on $[3, 4]$. Therefore f^3 has a unique fixed point,
which is also a fixed point of f .

Period 5 orbit, but no period 3 orbit

Example. $n = 5$, $\pi = (13425)$.



Theorem The map f is chaotic on the interval $[1, 5]$.

Lemma The map f is expansive on the interval $[1, 5]$.