MATH 614

Dynamical Systems and Chaos

Lecture 25:
Chain recurrence.

Chain recurrence

Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

Definition. A point $x \in X$ is **recurrent** for the map F if for any $\varepsilon > 0$ there is an integer $n > 0$ such that $d\big(F^n(x), x\big) < \varepsilon$. The point x is **chain recurrent** for F if, for any $\varepsilon > 0$, there are points $x_0 = x, x_1, x_2, \ldots, x_k = x$ and positive integers $n_1, n_2, \ldots, n_k$ such that $d\big(F^{n_i}(x_{i-1}), x_i\big) < \varepsilon$ for $1 \leq i \leq k$.

A sequence $x_0, x_1, \ldots, x_k$ is called an ε-**pseudo-orbit** of the map F if $d\big(F(x_{i-1}), x_i\big) < \varepsilon$ for $1 \leq i \leq k$. The point $x \in X$ is chain recurrent for F if, for any $\varepsilon > 0$, there exists an ε-pseudo-orbit $x_0, x_1, \ldots, x_k$ with $x_0 = x_k = x$.

Chain recurrence: properties

- Any periodic point is recurrent.
- Any eventually periodic (but not periodic) point is not recurrent.
- If a point $x \in X$ is chain recurrent under a map $f : X \to X$, then so is $F(x)$.
- Any limit point of any orbit $x, F(x), F^2(x), \ldots$ is chain recurrent. In the case F is invertible, any limit point of any backward orbit $x, F^{-1}(x), F^{-2}(x), \ldots$ is chain recurrent.
- If the orbit of x is dense in X, then x is recurrent unless x is an isolated point in X and not periodic for F.
- The set of all chain recurrent points is closed.
- For a topologically transitive map, all points are chain recurrent.
- Topological conjugacy preserves recurrence and chain recurrence.

- If $x \in W^s(p)$ for a periodic point p, then x is not recurrent unless $x = p$.

- If $X = S^1$ and F is a rotation then every point is recurrent (since either all points are periodic or all orbits are dense).

- If X is the torus $\mathbb{T}^n$ and F is a translation then every point is recurrent (since F preserves distances and volume).

- If $X = \Sigma_{\mathcal{A}}$ and $F = \sigma$ is the one-sided shift, then every point $\mathbf{s} \in X$ is chain recurrent. Indeed, let $\mathbf{s}^{(n)} = w_n w_n w_n \ldots,$ where w_n is the beginning of $\mathbf{s}$ of length n. Then $\sigma^n\big(\mathbf{s}^{(n)}\big) = \mathbf{s}^{(n)}$ and $\mathbf{s}^{(n)} \to \mathbf{s}$ as $n \to \infty$.

- If $X = \Sigma_{\mathcal{A}}$ and $F = \sigma$ is the one-sided shift, then not every point is recurrent. For example, $\mathbf{s} = (1000\ldots)$ is not recurrent.

- If $X = \Sigma_{\mathcal{A}}^{\pm}$ and $F = \sigma$ is the two-sided shift, then every point is chain recurrent but not every point is recurrent, e.g., $\mathbf{s} = (\ldots 000.1000\ldots)$.

Let $F : X \to X$ be a homeomorphism of a metric space X.

Definition. Suppose $x \in W^s(p) \cap W^u(q)$, where p and q are periodic points of F. Then x is called **heteroclinic** if $p \neq q$ and **homoclinic** if $p = q$.

- Any homoclinic point is chain recurrent.

- If $X = \mathbb{T}^2$ and F is a hyperbolic toral automorphism, then all points of X are chain recurrent (periodic points of F are dense and so are homoclinic points for the fixed point $[0, 0]$).

- If F is the logistic map $F(x) = \mu x(1 - x)$, $\mu > 4$, then chain recurrent points are all points of the invariant Cantor set.

- If F is the solenoid map, then chain recurrent points are all points of the solenoid.

- If F is the horseshoe map, then chain recurrent points are the attracting fixed point and all points of the invariant Cantor set.

Morse-Smale diffeomorphisms

Definition. A diffeomorphism $F : X \to X$ is called **Morse-Smale** if

(i) it has only finitely many chain recurrent points,

(ii) every chain recurrent point is periodic,

(iii) every periodic point is hyperbolic,

(iv) all intersections of stable and unstable manifolds of saddle points of F are transversal.

Theorem (Palis) Any Morse-Smale diffeomorphism of a compact surface is C^1-structurally stable.