MATH 614

Dynamical Systems and Chaos

Lecture 27:
Holomorphic dynamics.

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

$x =$ real part of z,
$iy =$ imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add, subtract, and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The **modulus** of z is $|z| = \sqrt{x^2 + y^2}$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Complex exponentials

Definition. For any $z \in \mathbb{C}$ let

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

Remark. A sequence of complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2, \ldots$ converges to $z = x + iy$ if $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Theorem 1 If $z = x + iy,\ x, y \in \mathbb{R}$, then

$$e^z = e^x(\cos y + i \sin y).$$

In particular, $e^{i\phi} = \cos\phi + i\sin\phi,\ \phi \in \mathbb{R}$.

Theorem 2 $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$.

Proposition $e^{i\phi} = \cos\phi + i\sin\phi$ for all $\phi \in \mathbb{R}$.

Proof: $e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \cdots + \frac{(i\phi)^n}{n!} + \cdots$

The sequence $1, i, i^2, i^3, \ldots, i^n, \ldots$ is periodic:
$\underbrace{1, i, -1, -i}, \underbrace{1, i, -1, -i}, \ldots$

It follows that

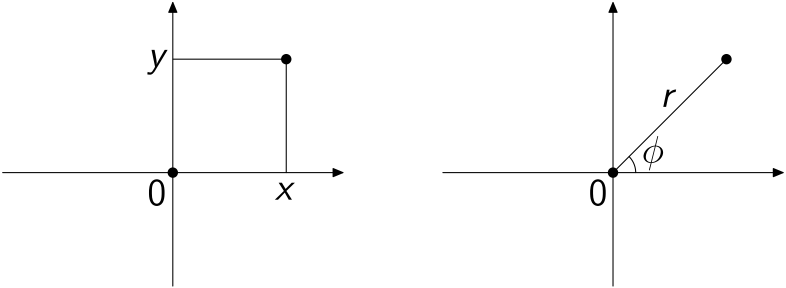
$$e^{i\phi} = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots + (-1)^k \frac{\phi^{2k}}{(2k)!} + \cdots$$

$$+ \; i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots + (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} + \cdots\right)$$

$$= \cos\phi + i\sin\phi.$$

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.



$$x = r\cos\phi, \;\; y = r\sin\phi \;\; \Longrightarrow \;\; z = r(\cos\phi + i\sin\phi) = re^{i\phi}$$

If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, then
$z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$, $z_1/z_2 = (r_1/r_2)e^{i(\phi_1-\phi_2)}$.

Fundamental Theorem of Algebra
Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Holomorphic functions

Suppose $D \subset \mathbb{C}$ is a domain and consider a function $f : D \to \mathbb{C}$. The function f is called **complex differentiable** at a point $z_0 \in D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

The limit value is the **derivative** $f'(z_0)$.

The function f is called **holomorphic at** a point $z_0 \in D$ if it is complex differentiable in a neighborhood of z_0. f is **holomorphic on** D if it is holomorphic at every point of D.

To each complex function $f : D \to \mathbb{C}$ we associate a real vector-valued function $(u, v) : D \to \mathbb{R}^2$ defined by $f(x + iy) = u(x, y) + iv(x, y)$.

Theorem The function f is holomorphic if and only if u, v have continuous partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and, moreover, the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Analytic functions

The function $f : D \to \mathbb{C}$ is called **analytic at** a point $z_0 \in D$ if it can be expanded into a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

in a neighborhood of z_0. f is **analytic on** D if it is analytic at every point of D.

Examples.

- Any complex polynomial is an analytic function on $\mathbb{C}$.
- Any rational function $R(z) = P(z)/Q(z)$, where P, Q are polynomials, is analytic on its domain.
- The exponential function is analytic on $\mathbb{C}$.

Theorem A function $f : D \to \mathbb{C}$ is analytic on D if and only if it is holomorphic on D. If f is analytic then it coincides with its Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

on any open disk $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ that is contained within D.

Complex linear functions

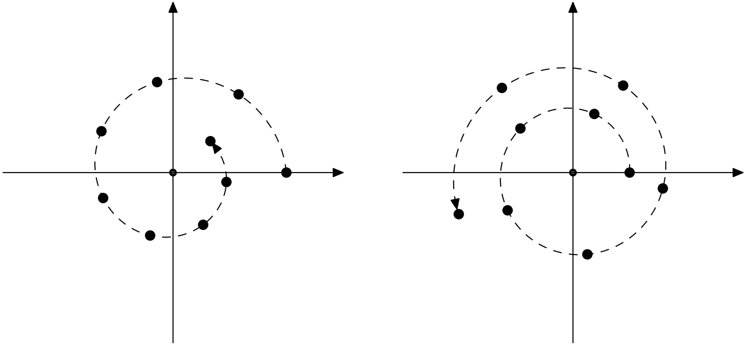
$L_\alpha : \mathbb{C} \to \mathbb{C},\ \alpha \in \mathbb{C}.$
$L_\alpha(z) = \alpha z$ for all $z \in \mathbb{C}.$

If $\alpha = 1$ then L_α is the identity map. Otherwise 0 is the only fixed point.

Dynamics of L_α depends on α.
$L_\alpha^n(z) = \alpha^n z$ for $n = 1, 2, \ldots$
Let $\alpha = \rho e^{i\theta},\ z = re^{i\phi}$. Then

$$L_\alpha^n(z) = \rho^n r e^{i(n\theta+\phi)}.$$

If $|\alpha| < 1$ then $\lim_{n\to\infty} L_\alpha^n(z) = 0$ for all $z \in \mathbb{C}.$
If $|\alpha| > 1$ then $\lim_{n\to\infty} L_\alpha^n(z) = \infty$ for all $z \neq 0.$



$|\alpha| = 0.9$ $|\alpha| = 1.1$

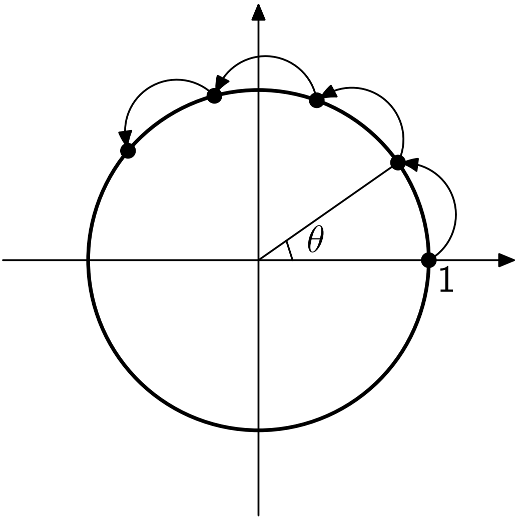
Rotations of the plane

If $|\alpha| = 1$ then L_α is the rotation of the complex plane by angle θ, the argument of α $(\alpha = e^{i\theta})$.

Each circle $\{z \in \mathbb{C} : |z| = r\}$, $r > 0$ is invariant under L_α. The restriction of L_α is a rotation of the circle.

In polar coordinates (r, ϕ),

$$(r, \phi) \mapsto (r, \phi + \theta).$$


θ
1

The argument of α, $|\alpha| = 1$ is a rational multiple of π if and only if α is a root of unity: $\alpha^k = 1$ for some integer $k > 0$.

If α is a root of unity $\sqrt[k]{1}$, then L_α^k is the identity. Hence all orbits are periodic.

If α is not a root of unity then
(i) each orbit is dense in a circle centered at the origin (Jacobi's Theorem);
(ii) each orbit is uniformly distributed with respect to the length measure on the circle
(the Kronecker-Weyl Theorem).

Complex affine functions

$L_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}, \ \ \alpha, \beta \in \mathbb{C}$.
$L_{\alpha,\beta}(z) = \alpha z + \beta$ for all $z \in \mathbb{C}$.

$L_{1,\beta}$ is the translation of the complex plane by β.
$L_{1,\beta}^n(z) = z + n\beta$ for $n = 1, 2, \dots$
Each orbit tends to infinity (unless $\beta \neq 0$).

If $\alpha \neq 1$ then $L_{\alpha,\beta}$ is conjugate to L_α.

The equation $L_{\alpha,\beta}(z) = z$ has a unique solution $z_0 = \beta(1-\alpha)^{-1}$. Then $L_{\alpha,\beta}(z) - z_0 = L_\alpha(z - z_0)$ for all $z \in \mathbb{C}$.

Hence $L_{\alpha,\beta} = L_{1,z_0} L_\alpha L_{1,z_0}^{-1}$.

Squaring function

$Q_0 : \mathbb{C} \to \mathbb{C}, \;\; Q_0(z) = z^2.$

Let $z = re^{i\phi}$. Then $Q_0(z) = r^2 e^{2i\phi}$.

$Q_0^n(z) = z^{2^n} = r^{2^n} e^{i(2^n\phi)}$.

If $r = |z| < 1$ then $Q_0^n(z) \to 0$ as $n \to \infty$.
If $|z| > 1$ then $Q_0^n(z) \to \infty$ as $n \to \infty$.

The unit circle $|z| = 1$ is invariant under Q_0 and the restriction of Q_0 is conjugate to the doubling map.

In polar coordinates (r, ϕ),

$$(r, \phi) \mapsto (r^2, 2\phi).$$

Theorem The squaring map Q_0 is chaotic on the unit circle, that is,

- it is topologically transitive,
- periodic points are dense,
- it has sensitive dependence on initial conditions.

Proposition For any $z \in \mathbb{C}$, $|z| = 1$ and any neighborhood W of z we have

$$\bigcup_{n=0}^{\infty} Q_0^n(W) = \mathbb{C} \setminus \{0\}.$$

Proof: Any neighborhood of a point on the unit circle contains a small chunk of a wedge of the form

$$V = \{re^{i\phi} \mid r_1 < r < r_2,\ \phi_1 < \phi < \phi_2\},$$

where $r_1 < 1 < r_2$. Now

$$Q_0^n(V) = \{re^{i\phi} \mid r_1^{2^n} < r < r_2^{2^n},\ 2^n\phi_1 < \phi < 2^n\phi_2\}$$

for $n = 1, 2, \ldots$ If $2^n(\phi_2 - \phi_1) > 2\pi$ then

$$Q_0^n(V) = \{z \in \mathbb{C} : r_1^{2^n} < |z| < r_2^{2^n}\}.$$

Since $r_1 < 1 < r_2$, it follows that

$$\bigcup_{n=0}^{\infty} Q_0^n(V) = \mathbb{C} \setminus \{0\}.$$

Fixed points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a holomorphic function.

Suppose that $F(z_0) = z_0$ for some $z_0 \in U$.

The fixed point z_0 is called

- attracting if $|F'(z_0)| < 1$;
- repelling if $|F'(z_0)| > 1$;
- neutral if $|F'(z_0)| = 1$.

Example. $L'_\alpha(0) = \alpha$.

Theorem 1 Suppose z_0 is an attracting fixed point for a holomorphic function F. Then there exist $\delta > 0$ and $0 < \mu < 1$ such that

$$|F(z) - z_0| \leq \mu |z - z_0|$$

for any $z \in D = \{z \in \mathbb{C} : |z - z_0| < \delta\}$.

In particular, $\lim_{n \to \infty} F^n(z) = z_0$ for all $z \in D$.

Hint. Take $|F'(z_0)| < \mu < 1$.

Theorem 2 Suppose z_0 is a repelling fixed point for a holomorphic function F. Then there exist $\delta > 0$ and $M > 1$ such that

$$|F(z) - z_0| \geq M|z - z_0|$$

for all $z \in D = \{z \in \mathbb{C} : |z - z_0| < \delta\}$.

In particular, for any $z \in D \setminus \{z_0\}$ there is an integer $n > 0$ such that $F^n(z) \notin D$.

Hint. Take $1 < M < |F'(z_0)|$.

Periodic points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to U$ be a holomorphic function. Suppose that $F^n(z_0) = z_0$ for some $z_0 \in U$ and an integer $n > 0$.

The periodic orbit
$z_0, F(z_0), F^2(z_0), \ldots, F^{n-1}(z_0), F^n(z_0) = z_0, \ldots$
is called

- attracting if $|(F^n)'(z_0)| < 1$;
- repelling if $|(F^n)'(z_0)| > 1$;
- neutral if $|(F^n)'(z_0)| = 1$.

$$(F^n)'(z_0) = \prod_{k=0}^{n-1} F'(F^k(z_0)).$$