MATH 614

Dynamical Systems and Chaos

Lecture 33:
The Julia and Fatou sets (continued).

The Julia and Fatou sets

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P. The **Fatou set** $S(P)$ of P is the set of all points $z \in U$ such that the family of iterates $\{P^n\}_{n\geq 1}$ is normal at z.

- The Julia set is closed, the Fatou set is open.
- The Julia and Fatou sets are disjoint.
- Attracting periodic points of P belong to $S(P)$.
- $P(J(P)) \subset J(P)$.
- If $U = \overline{\mathbb{C}}$, then $P(J(P)) = J(P)$.
- $P(S(P)) \subset S(P)$ and $P^{-1}(S(P)) \subset S(P)$. In fact, $P^{-1}(S(P)) = S(P)$.

Montel's Theorem

Theorem (Montel) Suppose $\mathcal{F}$ is a family of holomorphic functions defined on a domain $U \subset \mathbb{C}$. If the functions from $\mathcal{F}$ do not assume two values $a, b \in \mathbb{C}$, then $\mathcal{F}$ is a normal family in U.

Corollary 1 If $P : U \to U$ is a holomorphic map, where $U \subset \mathbb{C}$ and $\mathbb{C} \setminus U$ contains at least two points, then $S(P) = U$ and $J(P) = \emptyset$.

Corollary 2 Suppose $z \notin S(P)$ and W is a neighborhood of z. Then $\bigcup_{n=1}^{\infty} P^n(W)$ is either $\mathbb{C}$ or $\mathbb{C}$ minus one point.

More properties of the Julia and Fatou sets

- If the Fatou set is not empty, then the Julia set has empty interior.

- There exists a rational function P such that $J(P) = \overline{\mathbb{C}}$ and $S(P) = \emptyset$.

- If $P(z) = \exp z$, then $J(P) = \mathbb{C}$ and $S(P) = \emptyset$.

- If the Julia set is more than one repelling orbit, then it has no isolated points.

- $J(P^n) = J(P)$ for all $n \geq 1$.

Homoclinic points

Let z_0 be a repelling fixed point of a holomorphic map P.

Suppose $z_{-1}, z_{-2}, \ldots$ is a sequence of points such that $P(z_k) = z_{k+1}$ for $k = -1, -2, \ldots$ and $z_{-n} \to z_0$ as $n \to \infty$.

Then the points $z_{-1}, z_{-2}, \ldots$ are called **homoclinic** for z_0.

Theorem Homoclinic points belong to the Julia set $J(P)$.

More properties of the Julia and Fatou sets

- The union of the Julia and Fatou sets of P is the entire domain of P.
- $P(J(P)) = J(P)$.
- $P^{-1}(J(P)) = J(P)$.
- For any repelling fixed point z_0 of P, the homoclinic points for z_0 are dense in $J(P)$.
- For any $z_0 \in J(P)$, the Julia set $J(P)$ is the closure of the set $\bigcup_{n \geq 0} P^{-n}(z_0)$.

Dynamics on the Julia set

Proposition 1 The restriction of a holomorphic map P to its Julia set $J(P)$ is topologically transitive.

Proposition 2 If the Julia set $J(P)$ consists of more than one repelling orbit, then the map P has sensitive dependence on initial conditions on $J(P)$.

Theorem If the Julia set $J(P)$ consists of more than one repelling orbit, then the map P is chaotic on $J(P)$.

Proposition 1 The restriction of a holomorphic map P to its Julia set $J(P)$ is topologically transitive.

Proof: We need to show that for any nonempty open sets $U_1, U_2 \subset J(P)$ there exists $n \geq 1$ such that $P^n(U_1) \cap U_2 \neq \emptyset$.

Here $U_1 = W_1 \cap J(P)$, $U_2 = W_2 \cap J(P)$, where W_1, W_2 are open sets in $\mathbb{C}$.

We know that $\bigcup_{n\geq 1} P^n(W_1)$ is $\mathbb{C}$ or $\mathbb{C}$ minus one point. It follows that $P^n(W_1) \cap U_2 \neq \emptyset$ for some n. But $P^n(W_1) \cap U_2 = P^n(U_1) \cap U_2$.

Proposition 2 If the Julia set $J(P)$ consists of more than one repelling orbit, then the map P has sensitive dependence on initial conditions on $J(P)$.

Proof: We need to find $\beta > 0$ such that for any $z_0 \in J(P)$ and any neighborhood U of z_0 (in $J(P)$) we have $|P^n(z) - P^n(z_0)| \geq \beta$ for some $n \geq 1$ and $z \in U$.

By assumption, the Julia set contains two different repelling periodic orbits: $z_1, z_2, \dots, z_m$ and $w_1, w_2, \dots, w_k$. Choose $\beta > 0$ so that $|z_j - w_l| > 2\beta$ for all j and l.

Let $z_0 \in J(P)$ and U be a neighborhood of z_0. We know that $\bigcup_{n \geq 1} P^n(U) = J(P)$ or $J(P)$ minus one point. In the latter case, the one point is not a repelling fixed point. Hence we can find $z, w \in U$ such that $P^{n_1}(z) = z_1$ and $P^{n_2}(w) = w_1$ for some $n_1, n_2 \geq 1$. Now take any $n \geq \max(n_1, n_2)$. Then $P^n(z)$ is in the cycle $z_1, z_2, \dots, z_m$ while $P^n(w)$ is in the cycle $w_1, w_2, \dots, w_k$. In particular, $|P^n(z) - P^n(w)| > 2\beta$. It follows that $|P^n(z) - P^n(z_0)| > \beta$ or $|P^n(w) - P^n(z_0)| > \beta$.