MATH 614

Dynamical Systems and Chaos

Lecture 35:
The filled Julia set (continued).
More on holomorphic dynamics.

The filled Julia set

Definition. The **filled Julia set** of the polynomial P, denoted $K(P)$, is the set of all points $z \in \mathbb{C}$ such that the orbit $z, P(z), P^2(z), \dots$ is bounded.

Proposition 1 The complement of $K(P)$ consists of points whose orbits escape to infinity.

Proposition 2 There is $R_0 > 0$ such that the set $\{z \in \mathbb{C} : |z| > R_0\}$ is contained in the Fatou set.

Proposition 3 The Julia set and the filled Julia set are bounded.

Proposition 4 The Julia set is contained in the filled Julia set.

More properties of the filled Julia set

- The filled Julia set is completely invariant: $P(K(P)) \subset K(P)$ and $P^{-1}(K(P)) \subset K(P)$.

- The complement of the filled Julia set is contained in the Fatou set.

- The filled Julia set is closed.

- The filled Julia set is nonempty.

- The interior of the filled Julia set is contained in the Fatou set.

More properties of the filled Julia set

Proposition The boundary of the filled Julia set is disjoint from the Fatou set.

Proof: Suppose $z \in \partial K(P)$ and U is an arbitrary neighborhood of z. Then there are points $z_1, z_2 \in U$ such that $z_1 \in K(P)$ while $z_2 \notin K(P)$. We have $|P^n(z_1)| < R < \infty$ while $P^n(z_2) \to \infty$ as $n \to \infty$. It follows that the family $P, P^2, P^3, \ldots$ is not normal in U.

Corollary The boundary of the filled Julia set is the complement of the Fatou set.

Theorem The Julia set is the boundary of the filled Julia set.

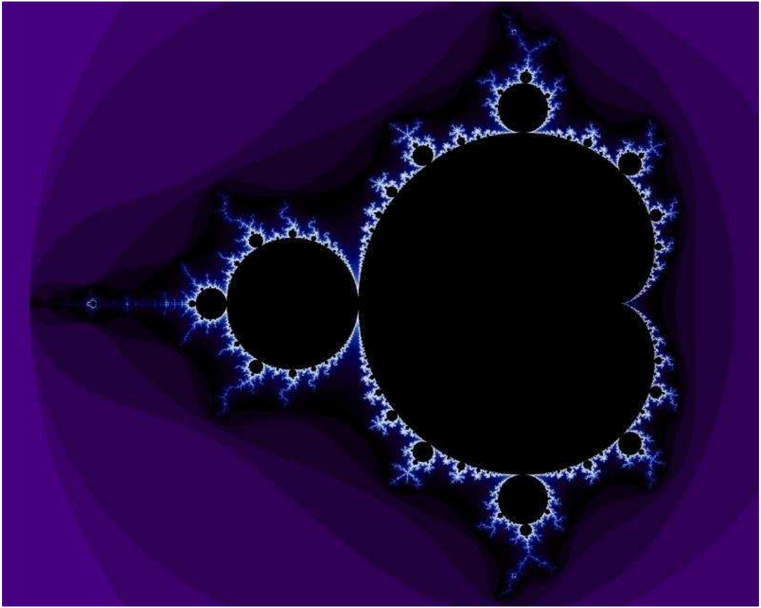
The Mandelbrot set

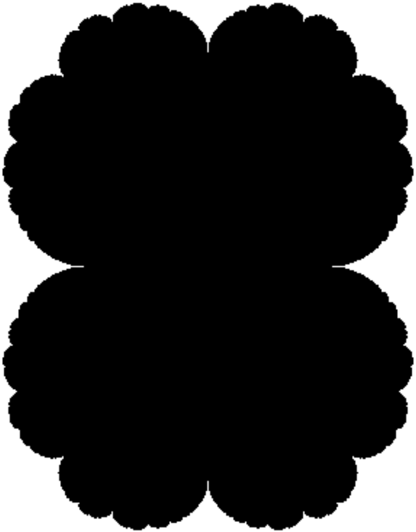
The quadratic family $Q_c : \mathbb{C} \to \mathbb{C},\ c \in \mathbb{C}$,
$Q_c(z) = z^2 + c$.

Theorem (Fundamental Dichotomy)
For any $c \in \mathbb{C}$,
either the post-critical orbit $0, Q_c(0), Q_c^2(0), \ldots$ escapes to ∞, in which case the Julia set $J(Q_c)$ is a Cantor set,
or the post-critical orbit is bounded, in which case the Julia set $J(Q_c)$ is connected.

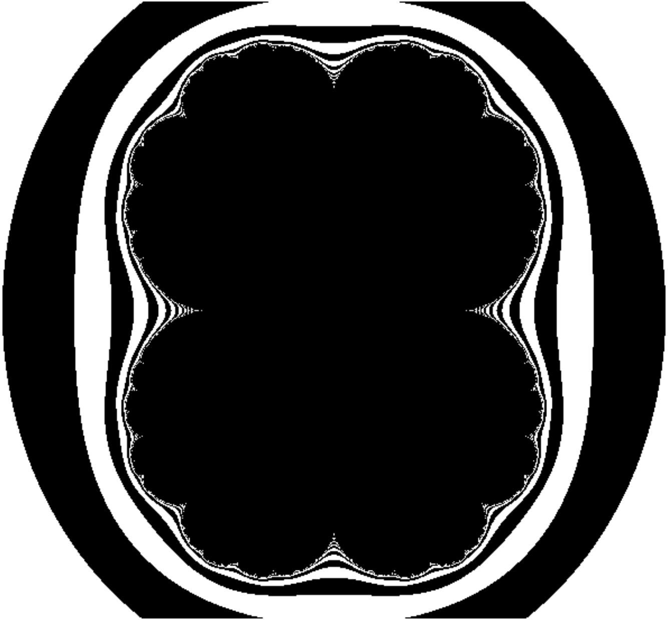
Definition. The **Mandelbrot set** $\mathcal{M}$ is the set of all $c \in \mathbb{C}$ such that $|Q_c^n(0)| \not\to \infty$ as $n \to \infty$.

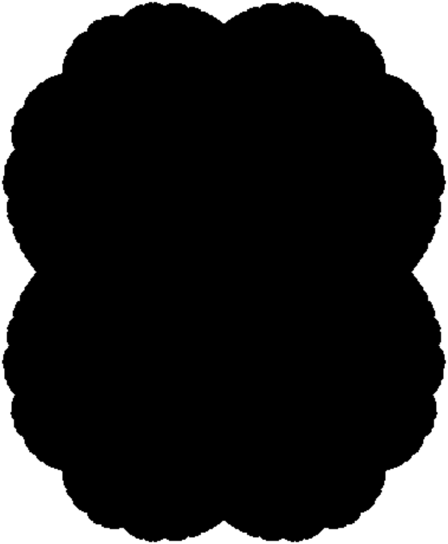
The Mandelbrot set $\mathcal{M}$ is the bifurcation diagram for the quadratic family.



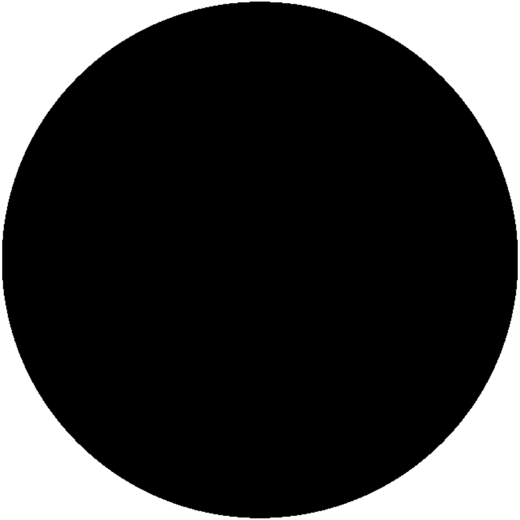


$K(Q_c),\ c = 0.25$

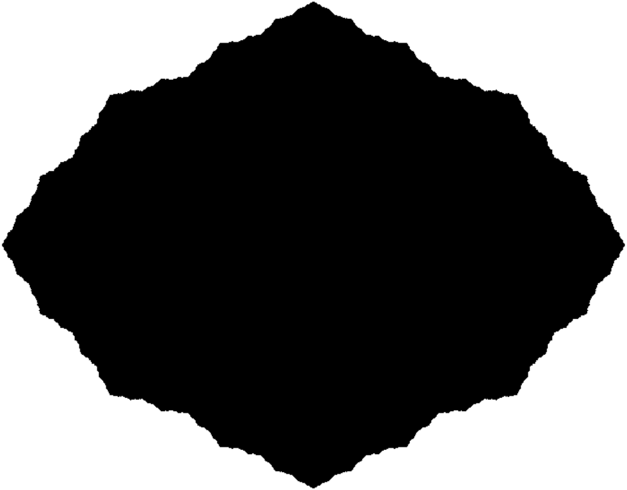




$K(Q_c),\ c = 0.2$

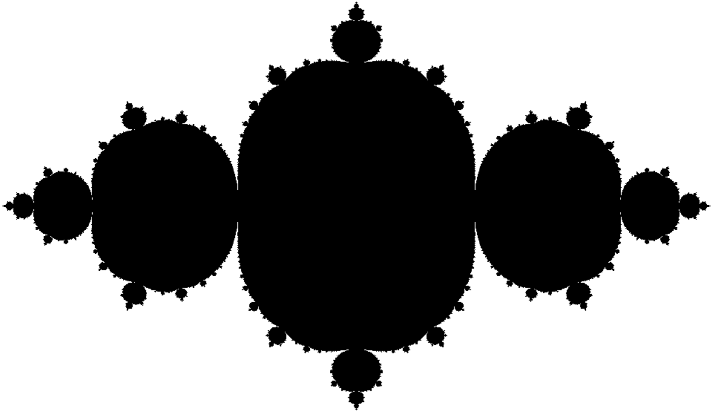


$K(Q_c),\ c = 0$

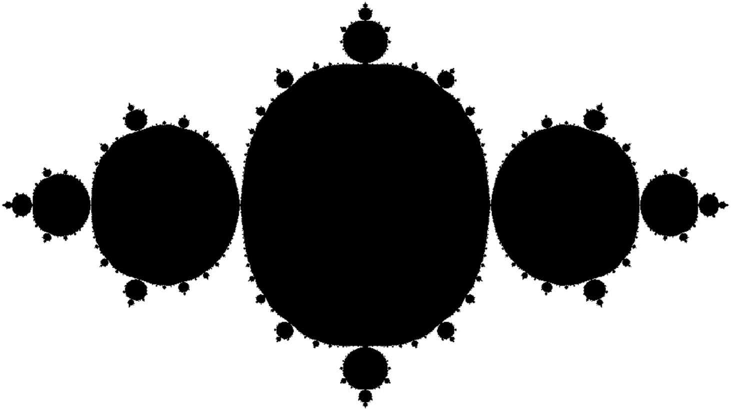


$K(Q_c), \ c = -0.3$

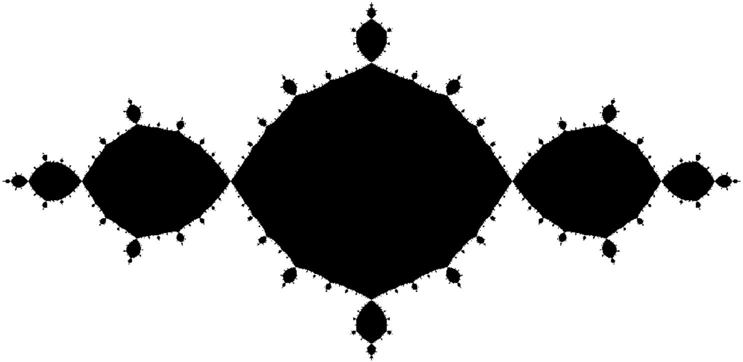
$K(Q_c),\ c = -0.6$



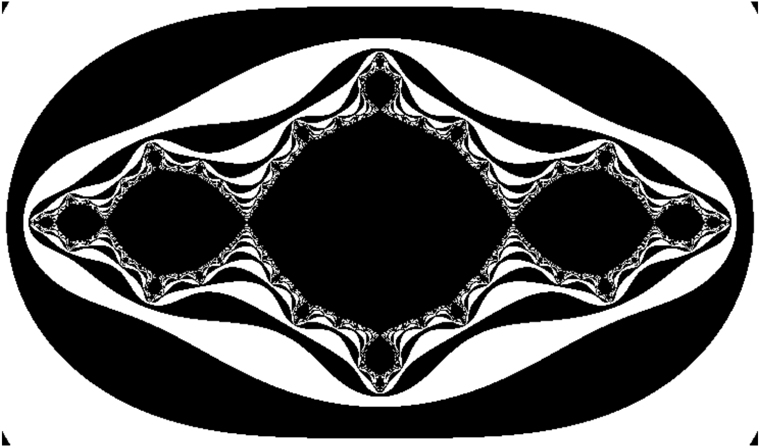
$K(Q_c),\ c = -0.75$

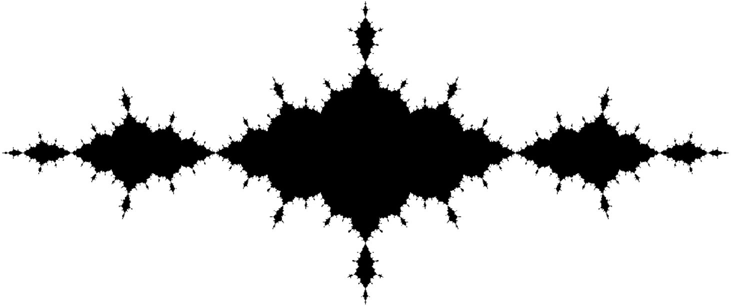


$K(Q_c),\ c = -0.8$

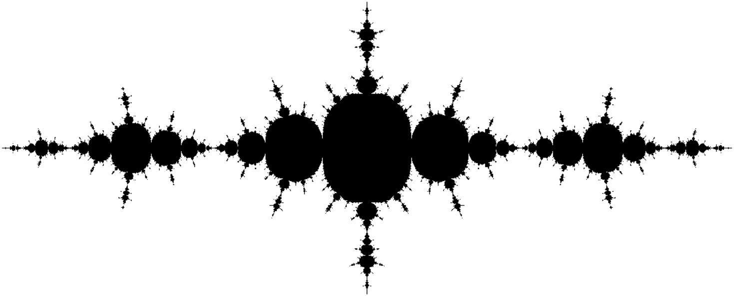


$K(Q_c),\ c = -1$

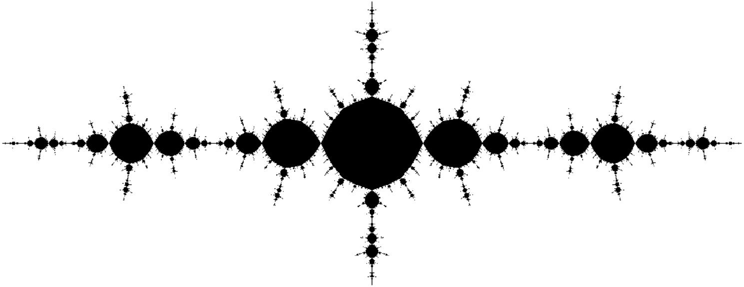




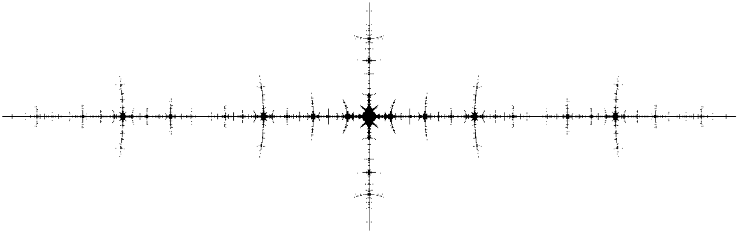
$K(Q_c),\ c = -1.2$



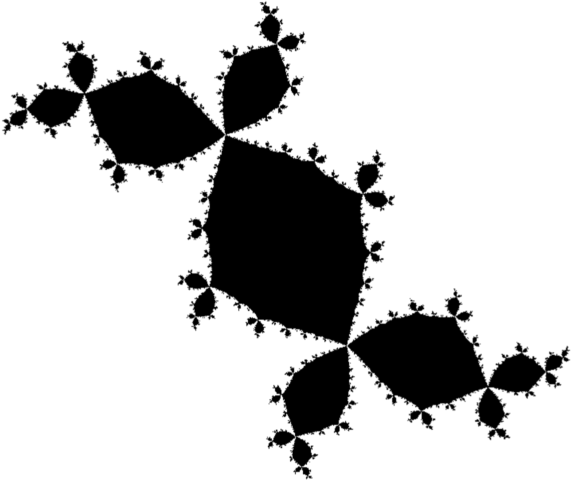
$K(Q_c),\ c = -1.25$



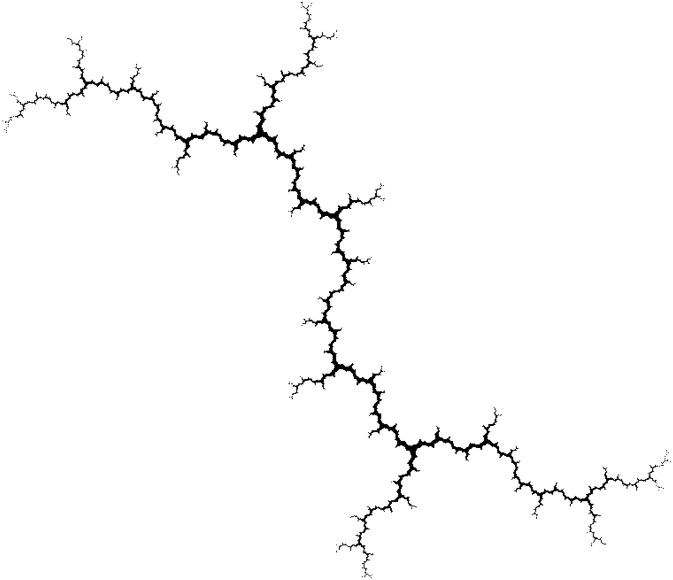
$K(Q_c),\ c = -1.3$



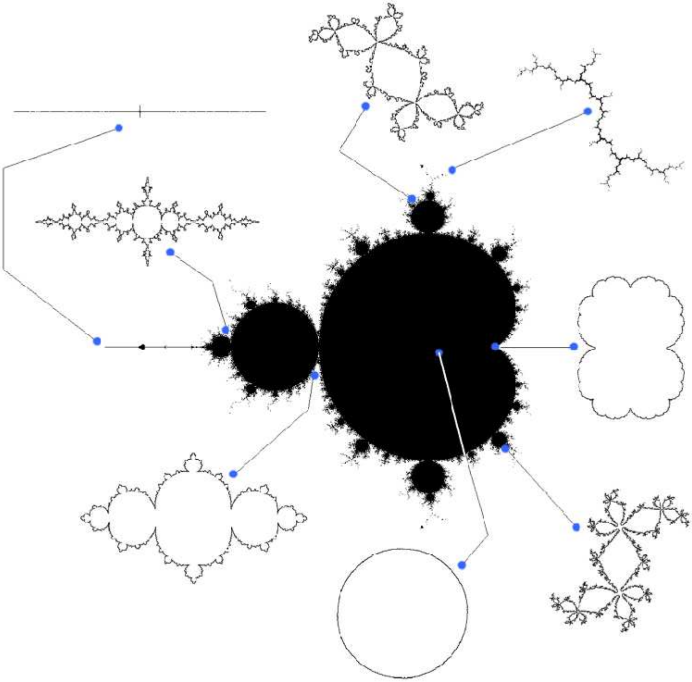
$K(Q_c), \ c = -1.5$

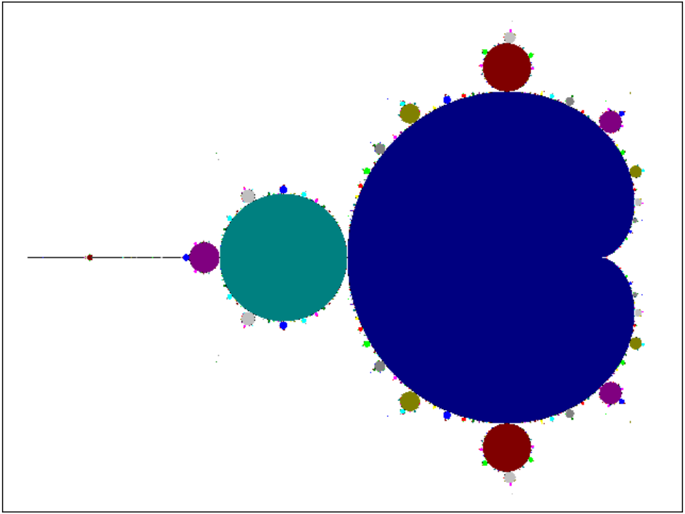


$K(Q_c),\ c = -0.122 + 0.745i$ ("Rabbit")



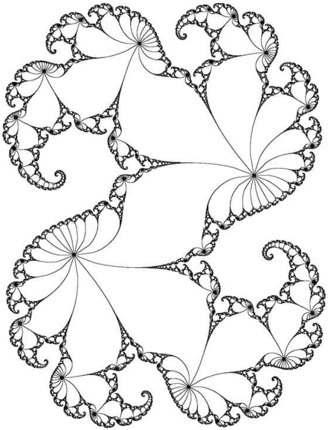
The Julia set of $z \mapsto z^2 + i$ ("Dendrite")





Period bulbs of the Mandelbrot set

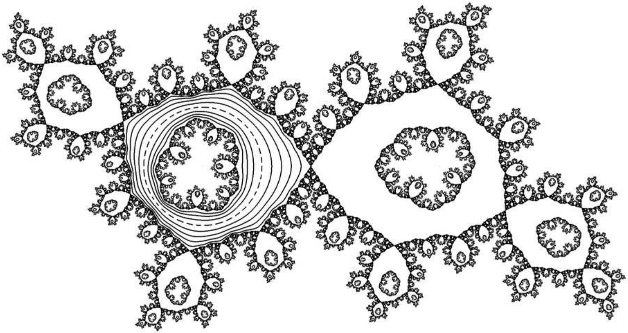
Attracting petals



The Julia set $J(Q_c)$, where $c \approx 0.29 + 0.176i$.

c is chosen on the boundary of the main cardioid so that Q_c has a fixed point with multiplier $\exp(\frac{2\pi i}{15})$.

Herman ring



The Julia set of $R(z) = e^{2\pi i\tau} z^2 \dfrac{z-4}{1-4z}$, $\tau \approx 0.615$.

The dashed curve is the unit circle $|z| = 1$, which is invariant under R. The restriction of R is an orientation-preserving homeomorphism. τ is chosen so that the rotation number is $(\sqrt{5}-1)/2$.