MATH 614

Dynamical Systems and Chaos

Lecture 38:
Ergodicity (continued).
Mixing.

Ergodic theorems

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

Birkhoff's Ergodic Theorem For any function $f \in L_1(X, \mu)$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f^*(x)$$

exists for almost all $x \in X$. The function f^* is T-invariant, i.e., $f^* \circ T = f^*$ almost everywhere.

von Neumann's Ergodic Theorem For any function $f \in L_2(X, \mu)$, the above limit exists in the Hilbert space $L_2(X, \mu)$. Moreover, f^* is the orthogonal projection of f on the subspace of functions invariant under Koopman's operator $U : L_2(X, \mu) \to L_2(X, \mu)$, $Uf = f \circ T$.

Remark. If $f \in L_1(X, \mu) \cap L_2(X, \mu)$, then the limit function f^* is the same in both theorems.

Ergodicity

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

Definition. The transformation T is called **ergodic** with respect to μ if any T-invariant measurable set E has either zero or full measure: $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Theorem The following conditions are equivalent:

- T is ergodic;
- for any sets $A, B \subset X$ of positive measure there exists an integer $n > 0$ such that $T^n(A) \cap B \neq \emptyset$;
- for any sets $A, B \subset X$ of positive measure there exists an integer $n > 0$ such that $\mu(A \cap T^{-n}(B)) > 0$;
- any measurable function $f : X \to \mathbb{C}$ invariant under T (that is, $f \circ T = f$ almost everywhere) is constant μ-a.e.;
- any function $f \in L_2(X, \mu)$ invariant under T is constant almost everywhere.

Rotations of the circle

Measured space $(S^1, \mathcal{B}(S^1), \mu)$, where μ is the length measure on S^1.

R_α: rotation of the unit circle by angle α.
R_α is a measure-preserving homeomorphism.

Theorem If α is not commensurable with π, then the rotation R_α is ergodic.

Let U_α be the associated operator on $L_2(S^1, \mu)$.

Relative to the angular coordinate on S^1, elements of $L_2(S^1, \mu)$ are 2π-periodic functions on $\mathbb{R}$. The inner product is given by

$$(f, g) = \int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

The operator U_α acts as follows:

$$(U_\alpha f)(x) = f(x + \alpha), \quad x \in \mathbb{R}.$$

For any $m \in \mathbb{Z}$ let $h_m(x) = e^{imx}$, $x \in \mathbb{R}$. Then $h_m \in L_2(S^1, \mu)$ and $U_\alpha h_m = e^{im\alpha} h_m$ so that h_m is an eigenfunction of U_α. Note that $\{h_m\}_{m \in \mathbb{Z}}$ is an orthogonal basis of the Hilbert space $L_2(X, \mu)$. We say that U_α has **pure point spectrum**.

Any $f \in L_2(X, \mu)$ is uniquely expanded as

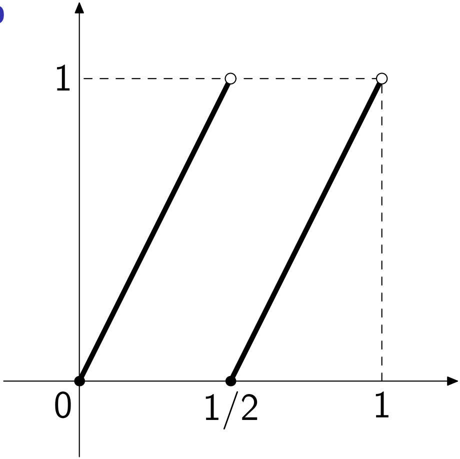
$$f = \sum\nolimits_{m \in \mathbb{Z}} c_m h_m, \quad \text{(Fourier series)}$$

where $c_m \in \mathbb{C}$. Then

$$U_\alpha f = \sum\nolimits_{m \in \mathbb{Z}} e^{im\alpha} c_m h_m.$$

Hence $U_\alpha f = f$ only if $(e^{im\alpha} - 1) c_m = 0$ for all $m \in \mathbb{Z}$. That is, if $f = c_0$, a constant.

Doubling map



$D : [0,1) \to [0,1)$,
$D(x) = 2x \bmod 1, \;\; x \in [0,1)$.

Theorem The doubling map is ergodic.

Sketch of the proof: We know that functions $h_n(x) = e^{inx}$, form an orthogonal basis of the Hilbert space $L_2(S^1, \mu)$. Koopman's operator U of the doubling map acts on them as follows:

$$h_0 \circlearrowleft$$

$$h_1 \longrightarrow h_2 \longrightarrow h_4 \longrightarrow h_8 \longrightarrow \cdots$$

$$h_3 \longrightarrow h_6 \longrightarrow h_{12} \longrightarrow h_{24} \longrightarrow \cdots$$

$$h_5 \longrightarrow h_{10} \longrightarrow h_{20} \longrightarrow h_{40} \longrightarrow \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Any $f \in L_2(X, \mu)$ is uniquely expanded into a Fourier series $f = \sum_{m \in \mathbb{Z}} c_m h_m$, where $c_m \in \mathbb{C}$. Then

$$Uf = \sum_{m \in \mathbb{Z}} c_m U(h_m) = \sum_{m \in \mathbb{Z}} c_m h_{2m}.$$

Hence $Uf = f$ only if $c_{2m} = c_m$ for all $m \in \mathbb{Z}$ and $c_m = 0$ for all odd integers m. That is, if $f = c_0$, a constant.

Theorem Any hyperbolic toral automorphism T_A of the flat torus is ergodic.

Proof: Let $f : \mathbb{T}^2 \to \mathbb{C}$ be a continuous function. By Birkhoff's Ergodic Theorem, for almost all $x \in \mathbb{T}^2$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_A^k(x)) = f^*(x),$$

where f^* is an integrable function. Also, for almost all $x \in \mathbb{T}^2$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_A^{-k}(x)) = f^{**}(x),$$

where f^{**} is an integrable function.

von Neumann's Ergodic Theorem implies that $f^* = f^{**}$ almost everywhere.

Let $x \in \mathbb{T}^2$ and $y \in W^s(x)$. Then $\mathrm{dist}(T_A^n(y), T_A^n(x)) \to 0$ as $n \to \infty$.
Since f is continuous, it follows that $|f(T_A^n(y)) - f(T_A^n(x))| \to 0$ as $n \to \infty$.
Therefore $f^*(y) = f^*(x)$.
Similarly, if $y \in W^u(x)$ then $f^{**}(y) = f^{**}(x)$.

Thus f^* is constant along leaves of the stable foliation while f^{**} is constant along leaves of the unstable foliation. Since $f^* = f^{**}$ a.e., it follows that f^* is constant almost everywhere.

Mixing

$(X, \mathcal{B}, \mu)$: measured space of finite measure
$T : X \to X$: measure-preserving transformation

T is called **mixing** if for any measurable sets $A, B \subset X$ we have

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \frac{\mu(A)\mu(B)}{\mu(X)}.$$

Lemma Mixing $\Longrightarrow$ ergodicity.

Proof: Suppose $C \subset X$ is measurable and T-invariant. Then $T^{-n}(C) = C$ up to a set of zero measure. Therefore $\mu(T^{-n}(C) \cap C) = \mu(C)$.

If T is mixing then $\mu(C) = \mu(C)\mu(C)/\mu(X)$, which implies that $\mu(C) = 0$ or $\mu(C) = \mu(X)$.

Theorem The doubling map is mixing.

Proof: Let $A \subset [0,1)$ and $n \geq 1$. Then $D^{-n}(A)$ is the union of 2^n disjoint pieces $\frac{1}{2^n}A + \frac{k}{2^n}$, $k = 0, 1, \ldots, 2^n - 1$.

Suppose $B = [\frac{l}{2^m}, \frac{l+1}{2^m})$, where $m > 0$, $0 \leq l < 2^m$. If $n \geq m$ then exactly 2^{n-m} pieces are contained in B, the others are disjoint from B. Hence

$$\mu(D^{-n}(A) \cap B) = 2^{n-m} \cdot 2^{-n}\mu(A) = \mu(A)\mu(B).$$

Since any measurable set B can be approximated by disjoint unions of the above intervals,

$$\lim_{n\to\infty} \mu(D^{-n}(A) \cap B) = \mu(A)\mu(B).$$

Proposition The rotation R_α of the circle is not mixing.

Proof: For any $\varepsilon > 0$ there exists $n > 0$ such that $R_\alpha^n = R_{n\alpha}$ is the rotation by an angle $< \varepsilon$.
Hence there exists a sequence $n_1 < n_2 < \dots$ such that for any arc $\gamma \subset S^1$,

$$\lim_{k\to\infty} \mu(R_\alpha^{-n_k}(\gamma) \cap \gamma) = \mu(\gamma).$$

But $\mu(\gamma) \neq \mu(\gamma)\mu(\gamma)/\mu(S^1)$ if $\gamma \neq S^1$.

$(X, \mathcal{B}, \mu)$: measured space of finite measure
$T : X \to X$: measure-preserving transformation

T is mixing if and only if for any $f, g \in L_2(X, \mu)$,

$$\lim_{n\to\infty} \int_X f(T^n(x))g(x)\, d\mu(x) = \frac{1}{\mu(X)} \int_X f\, d\mu \int_X g\, d\mu.$$

$$\lim_{n\to\infty} (U^n f, g) = \frac{(f,1)(1,g)}{(1,1)}.$$

Suppose f is a nonconstant eigenfunction of U, $Uf = \lambda f$, $|\lambda| = 1$. It is no loss to assume that $(f, 1) = 0$. Obviously,

$$(U^n f, f) = \lambda^n (f, f) \not\to 0 \quad \text{as} \quad n \to \infty.$$

$(X, \mathcal{B}, \mu)$: measured space of finite measure
$T : X \to X$: measure-preserving transformation
$U : L_2(X, \mu) \to L_2(X, \mu)$: associated linear operator

T is called **weakly mixing** if U has no eigenfunctions other than constants.

$$\text{mixing} \implies \text{weak mixing} \implies \text{ergodicity}$$

In particular, the doubling map has no nonconstant eigenfunctions. In this case, the operator U has **countable Lebesgue spectrum**. Namely, there are functions f_{nm} $(n, m = 1, 2, \dots)$ on S^1 such that 1 and f_{nm}, $n, m \geq 1$ form an orthogonal basis for $L_2(X, \mu)$, and $Uf_{nm} = f_{n,m+1}$ for any $n, m \geq 1$.

Translation of the torus $R_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$, $\alpha, \beta \in \mathbb{R}$.
$R_{\alpha,\beta}(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$.
$R_{\alpha,\beta}$ is a measure-preserving homeomorphism.

Theorem $R_{\alpha,\beta}$ has pure point spectrum. It is ergodic if and only if the numbers α, β, and 1 are linearly independent over $\mathbb{Q}$ (i.e., for any $k, m, n \in \mathbb{Z}$ the equality $k\alpha + m\beta + n = 0$ implies $k = m = n = 0$).

Doubling map $D_2 : \mathbb{T}^2 \to \mathbb{T}^2$;
$D_2(x_1, x_2) = (2x_1 \bmod 1, 2x_2 \bmod 1)$.

Theorem The doubling map on the torus preserves measure and is mixing. It has countable Lebesgue spectrum.