MATH 614

Dynamical Systems and Chaos

# Lecture 2:
# Periodic points.
# Hyperbolicity.

# Orbit

Let $f : X \to X$ be a map defining a discrete dynamical system. We use notation $f^n$ for the $n$-th iteration of $f$ defined inductively by $f^1 = f$ and $f^n = f^{n-1} \circ f$ for $n = 2, 3, \dots$.

*Definition.* The **forward orbit** (or simply **orbit**) of a point $x \in X$ under the map $f$ is a sequence $x, f(x), f^2(x), f^3(x), \dots$ The set of all points in this sequence is denoted $O^+(x)$ or $O_f^+(x)$. This set is also referred to as the **orbit** of $x$.

In the case $f$ is invertible, we can also define the **backward orbit** of $x$ under $f$ as the forward orbit of $x$ under the inverse map $f^{-1}$. Together, the forward and backward orbits form the **full orbit** of $x$ under $f$, which is a bi-infinite sequence

$$\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots$$

(here $f^{-n} = (f^{-1})^n$, $n = 2, 3, \dots$). The set of points in the backward orbit is denoted $O^-(x)$ and the set of points in the full orbit is denoted $O(x)$. Clearly, $O(x) = O^+(x) \cup O^-(x)$.

# Periodic points

*Definition.* A point $x \in X$ is called a **fixed** point of a map $f : X \to X$ if $f(x) = x$.

A point $x \in X$ is called a **periodic** point of a map $f : X \to X$ if $f^n(x) = x$ for some integer $n \geq 1$. The integer $n$ is called a **period** of $x$. The least integer $n$ satisfying this relation is called the **prime period** of $x$.

A point $x \in X$ is called an **eventually periodic** point of a map $f : X \to X$ if the orbit $O_f^+(x)$ is a finite set. In the case $f$ is invertible, any eventually periodic point is actually periodic.

# Examples

In all examples, $f$ is a transformation of the real line $\mathbb{R}$.

- $f(x) = -x$.

0 is a fixed point; the other points are periodic with period 2.

- $f(x) = \lambda x, \;\; 0 < |\lambda| < 1$.

0 is a fixed point; the orbit of any $x \neq 0$ converges to 0.

- $f(x) = x^3$.

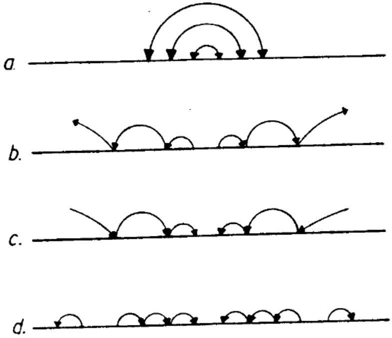
$-1$, 0, and 1 are fixed points. The orbit of any other $x \in \mathbb{R}$ is strictly monotone; it converges to 0 if $|x| < 1$ and diverges to infinity if $|x| > 1$.

- $f(x) = x^2$.

0 and 1 are fixed points while $-1$ is an eventually fixed point. The orbit of a point $x \in \mathbb{R}$ converges to 0 if $|x| < 1$ and diverges to infinity if $|x| > 1$.

# Phase portrait

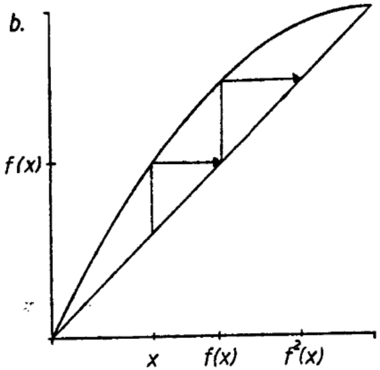
One-dimensional dynamical systems can be studied using the **phase portrait**, which is a picture of the real line with arrows joining some points to their images. In particular, an orbit is pictured as a chain of arrows.

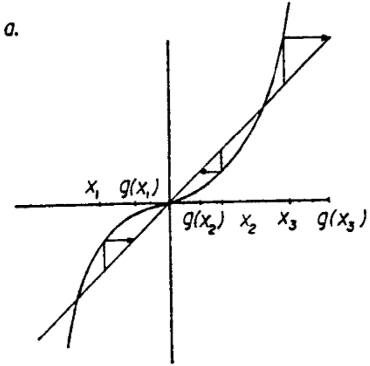



*Examples.* (a) $f(x) = -x$, (b) $f(x) = 2x$, (c) $f(x) = x/2$, (d) $f(x) = x^3$.

# Graphical analysis

Another way to study a one-dimensional dynamical systems $f$, called **graphic analysis**, uses the graph of the function $f$ along with the graph of the identity function.






This example shows graphic analysis of the function $g(x) = x^3$. It is typical for all continuous functions that are strictly increasing. Namely, every orbit is either increasing or decreasing or constant. In particular, the only periodic points are fixed points. Furthermore, every orbit either converges to a fixed point or diverges to infinity.

# Hyperbolicity

Suppose $X$ is an interval of the real line and $f : X \to X$ is a differentiable map.

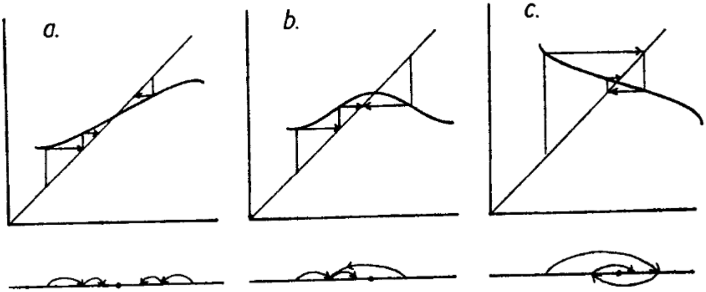
*Definition.* Let $p$ be a periodic point of $f$ of prime period $n$. The point $p$ is called **hyperbolic** if $|(f^n)'(p)| \neq 1$. The number $\lambda = (f^n)'(p)$ is called the **multiplier** of the periodic point $p$.

Note that $(f^n)'(p) = f'(p) \cdot f'(f(p)) \cdot \ldots \cdot f'(f^{n-1}(p))$. It follows that all periodic points in the same orbit have the same multiplier.

*Definition.* A hyperbolic periodic point with a multiplier $\lambda$ is called **repelling** if $|\lambda| > 1$ and **attracting** if $|\lambda| < 1$. It is called **super-attracting** if $\lambda = 0$.

# Attracting fixed points

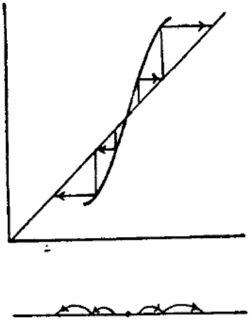
The phase portrait near a hyperbolic fixed point depends on its multiplier $\lambda$.



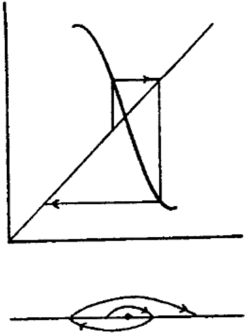

(a) $0 < \lambda < 1$, (b) $\lambda = 0$, (c) $-1 < \lambda < 0$.

## Repelling fixed points

The phase portrait near a hyperbolic fixed point depends on its multiplier $\lambda$.



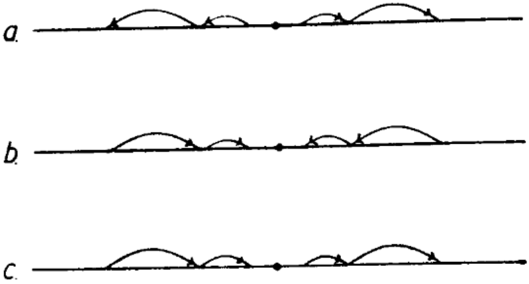
(a) $\lambda > 1$,



(b) $\lambda < -1$.

# Non-hyperbolic fixed points

In a small neighborhood of a non-hyperbolic fixed point, various dynamical systems can exhibit various kinds of behavior.

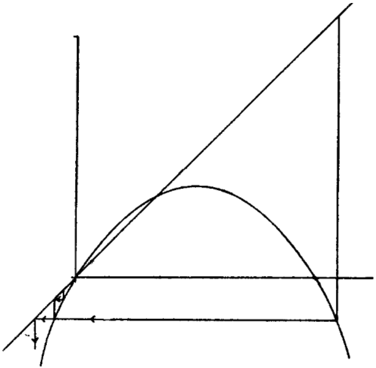



*Examples with the fixed point* $0$*:*

(a) $f(x) = x + x^3$, (b) $f(x) = x - x^3$, (c) $f(x) = x + x^2$.

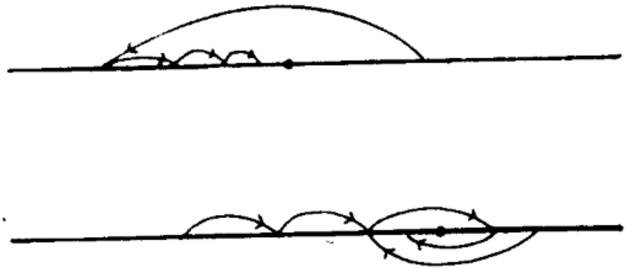
# Quadratic family

The **quadratic family** is the family of maps $F_\mu(x) = \mu x(1-x)$ depending on the parameter $\mu > 1$. The map $F_\mu$ has two fixed points 0 and $p_\mu = (\mu-1)/\mu \in (0,1)$. Besides, 1 is an eventually fixed point. Orbits of all points outside of the interval $[0,1]$ diverge to $-\infty$.



# Quadratic family

In the case $1 < \mu < 3$, the fixed point $p_\mu = (\mu - 1)/\mu$ is attracting. Moreover, the orbit of any point $x \in (0, 1)$ converges to $p_\mu$.



These are phase portraits of $F_\mu$ near the fixed point $p_\mu$ for $1 < \mu < 2$ and $2 < \mu < 3$.