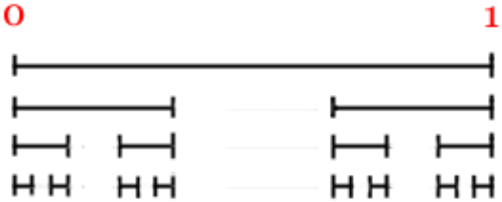
MATH 614

Dynamical Systems and Chaos

# Lecture 5:
# Cantor sets.
# Fractal dimension.
# Metric spaces.

# Cantor sets

Cantor Middle-Thirds Set




*Definition.* A subset $\Lambda$ of the real line $\mathbb{R}$ is called a (general) **Cantor set** if it is

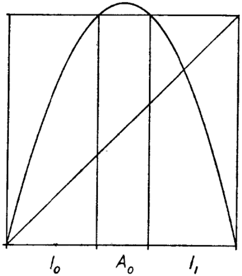
- nonempty,
- compact, which means that $\Lambda$ is bounded and closed,
- totally disconnected, which means that $\Lambda$ contains no intervals, and
- perfect, which means that $\Lambda$ has no isolated points.

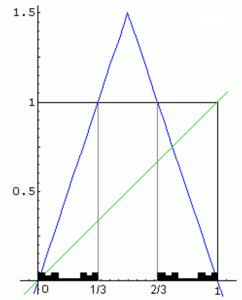
# Unimodal maps

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map such that

- $f(0) = f(1) = 0$;
- there exists a point $x_{\max} \in (0, 1)$ such that $f$ is strictly increasing on $(-\infty, x_{\max}]$ and strictly decreasing on $[x_{\max}, \infty)$;
- $f(x_{\max}) > 1$.

The map $f$ is called **unimodal**.



# Itinerary map

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, $\Lambda$ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$, and $S : \Lambda \to \Sigma_2$ be the **itinerary map** introduced in the previous lecture.

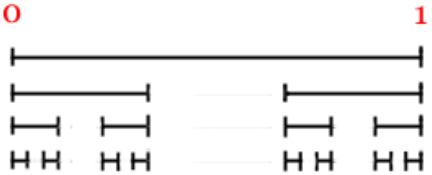
**Proposition 1** The set $\Lambda$ is compact and has no isolated points.

**Proposition 2** $S \circ f = \sigma \circ S$ on $\Lambda$, where $\sigma : \Sigma_2 \to \Sigma_2$ is the shift map.

**Proposition 3** The itinerary map $S$ is onto.

**Proposition 4** The set $\Lambda$ is a Cantor set if and only if the itinerary map $S$ is one-to-one.

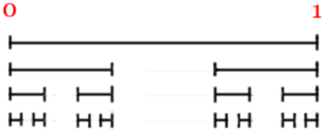
In the case $f$ is the tent map with $\mu = 3$, the interval $A_0$ is the middle third of $[0, 1]$ so that $\Lambda_3$ is exactly the Cantor Middle-Thirds Set.




The set $\Lambda_3$ consists of those points $x \in [0, 1]$ that admit a ternary expansion $0.s_1 s_2 \dots$ without any 1's (only 0's and 2's).

# Fractal dimension

The unit interval $[0, 1]$ is self-similar in the following sense. If you scale it by a factor of $n$ (where $n$ is a whole number), then it can be cut into $n$ unit intervals. Likewise, the unit square $[0, 1] \times [0, 1]$ is self-similar: if you scale it by a factor of $n$, then it can be cut into $n^2$ unit squares. Likewise, the unit box $[0, 1] \times [0, 1] \times [0, 1]$ is self-similar: if you scale it by a factor of $n$, then it can be cut into $n^3$ unit boxes.




The invariant Cantor set $\Lambda_\mu$ of the tent map $T_\mu$ $(\mu > 2)$ is self-similar as well. When you scale it by a factor of $\mu$, you get 2 copies of the original set. Scaling by a factor of $\mu^k$ produces $2^k$ copies of the original set.

Consequently, the dimension of $\Lambda_\mu$ is $\log_\mu 2 < 1$.

# General Cantor sets

*Definition.* A subset $\Lambda$ of the real line $\mathbb{R}$ is called a (general) **Cantor set** if it is

- nonempty,
- compact, which means that $\Lambda$ is bounded and closed,
- totally disconnected, which means that $\Lambda$ contains no intervals, and
- perfect, which means that $\Lambda$ has no isolated points.

**Theorem** Any two Cantor sets are homeomorphic.
That is, if $\Lambda$ and $\Lambda'$ are Cantor sets, then there exists a homeomorphism $\phi : \Lambda \to \Lambda'$ (an invertible map such that both $\phi$ and $\phi^{-1}$ are continuous).

Furthermore, the homeomorphism $\phi$ can be chosen strictly increasing, in which case it can be extended to a homeomorphism $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$.

An open subset $U \subset \mathbb{R}$ is a union of open intervals. An open interval $(a, b)$ is called a **maximal subinterval** of $U$ if there is no other interval $(c, d)$ such that $(a, b) \subset (c, d) \subset U$.

**Lemma 1** Any point of $U$ is contained in a maximal subinterval.

**Lemma 2** Finite endpoints of a maximal subinterval do not belong to $U$.

**Lemma 3** Distinct maximal subintervals are disjoint.

**Lemma 4** There are at most countably many maximal subintervals.

**Lemma 5** If $\Lambda$ is a Cantor set, then for any two maximal subintervals of $\mathbb{R} \setminus \Lambda$ there is another maximal subinterval that lies between them.

**Lemma 6** If $\Lambda, \Lambda'$ are Cantor sets sets then there exists a monotone one-to-one correspondence between maximal subintervals of their complements.

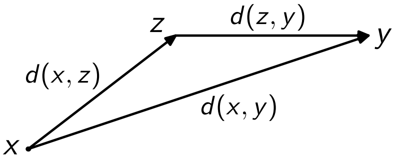
# Metric space

*Definition.* Given a nonempty set $X$, a **metric** (or **distance function**) on $X$ is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

• **(positivity)** $d(x, y) \geq 0$ for all $x, y \in X$; moreover, $d(x, y) = 0$ if and only if $x = y$;

• **(symmetry)** $d(x, y) = d(y, x)$ for all $x, y \in X$;

• **(triangle inequality)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.




A set endowed with a metric is called a **metric space**.

# Examples of metric spaces

- *Real line*

$X = \mathbb{R}$, $d(x, y) = |y - x|$.

- *Euclidean space*

$X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$.

- *Normed vector space*

$X$: vector space with a norm $\|\cdot\|$, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

- *Discrete metric space*

$X$: any nonempty set, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

- *Subspace of a metric space*

$X$: nonempty subset of a metric space $Y$ with a distance function $\rho : Y \times Y \to \mathbb{R}$, $d$ is the restriction of $\rho$ to $X \times X$.

# Convergence and continuity

Suppose $(X, d)$ is a metric space, that is, $X$ is a set and $d$ is a metric on $X$.

We say that a sequence of points $x_1, x_2, \ldots$ of the set $X$ **converges** to a point $y \in X$ if $d(x_n, y) \to 0$ as $n \to \infty$.

Given another metric space $(Y, \rho)$ and a function $f : X \to Y$, we say that $f$ is **continuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon$.

We say that the function $f$ is **continuous on a set** $U \subset X$ if it is continuous at each point of $U$.

# Space of infinite sequences

Let $\mathcal{A}$ be a finite set. We denote by $\Sigma_\mathcal{A}$ the set of all infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$. Elements of $\Sigma_\mathcal{A}$ are also referred to as **infinite words** over the **alphabet** $\mathcal{A}$.

For any infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$ and $\mathbf{t} = (t_1 t_2 \dots)$ in $\Sigma_\mathcal{A}$, let $d(\mathbf{s}, \mathbf{t}) = 2^{-n}$ if $s_i = t_i$ for $1 \le i \le n$ while $s_{n+1} \ne t_{n+1}$. Also, let $d(\mathbf{s}, \mathbf{t}) = 0$ if $s_i = t_i$ for all $i \ge 1$.

**Proposition** The function $d$ is a metric on $\Sigma_\mathcal{A}$.

Two infinite words are considered close in the metric space $(\Sigma_\mathcal{A}, d)$ if they have a long common beginning.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a unimodal map that admits an invariant Cantor set $\Lambda \subset [0, 1]$. Let $S : \Lambda \to \Sigma_2 = \Sigma_{\{0,1\}}$ be the itinerary map.

**Theorem** The itinerary map $S$ is continuous.