MATH 614

Dynamical Systems and Chaos

Lecture 12: Sharkovskii's theorem (continued).

Sharkovskii's Theorem

The **Sharkovskii ordering** is the following strict linear ordering of the natural numbers:

$$
\begin{array}{ccccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \dots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \dots \\
\rhd & 2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \dots \\
\end{array}
$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots \rhd 2^k \rhd \dots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Theorem 1 Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. If f admits a periodic point of prime period n and $n \rhd m$ for some $m \in \mathbb{N}$, then f admits a periodic point of prime period m as well.

Theorem 2 Suppose P is a set of natural numbers such that $n \in P$ and $n \rhd m$ imply $m \in P$ for all $m, n \in \mathbb{N}$. Then there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ with P as the set of prime periods of its periodic points.

Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. Given two closed bounded intervals $I_1, I_2 \subset J$, we write and draw $\boxed{I_1 \to I_2}$ if $f(I_1) \supset I_2$ (I_1 covers I_2 under action of f).

Lemma 1 If $I \to I$, then the interval I contains a fixed point of the map f.

Lemma 2 If the map f has a periodic orbit, then it has a fixed point.

Proof: Suppose x is a periodic point of f of prime period n. In the case $n = 1$, we are done. Otherwise let $x_1, x_2, \dots, x_n$ be the list of all points of the orbit $O_f^+(x)$ ordered so that $x_1 < x_2 < \cdots < x_n$. Note that $f(x_i) \neq x_i$ for all i. In particular, $f(x_1) > x_1$ while $f(x_n) < x_n$.

Let j be the largest index satisfying $f(x_j) > x_j$. Then $j < n$, $f(x_j) \geq x_{j+1}$, and $f(x_{j+1}) \leq x_j$. The Intermediate Value Theorem implies that $[x_j, x_{j+1}] \to [x_j, x_{j+1}]$. By Lemma 1, the map f has a fixed point in the interval $[x_j, x_{j+1}]$.

Lemma 3 If $I \to I'$, then there exists a closed interval $I_0 \subset I$ such that f maps I_0 onto I'.

Proof: Let $I' = [a, b]$. Then $A = I \cap f^{-1}(a)$ and $B = I \cap f^{-1}(b)$ are nonempty compact sets. It follows that the distance function $d(x, y) = |y - x|$ attains its minimum on the set $A \times B$ at some point (x_0, y_0). Note that $x_0 \neq y_0$ since $A \cap B = \emptyset$. Let I_0 denote the closed interval with endpoints x_0 and y_0. Then $I_0 \subset I$, the endpoints of I_0 are mapped to a and b, and no interior point of I_0 is mapped to a or b. The Intermediate Value Theorem implies that $f(I_0) = I'$.

Lemma 4 If $I_1 \to I_2 \to \cdots \to I_n \to I_1$, then there exists a fixed point x of f^n such that $x \in I_1$, $f(x) \in I_2$, ..., $f^{n-1}(x) \in I_n$.

Proof: It follows by induction from Lemma 3 that there exist closed intervals $I_1' \subset I_1$, $I_2' \subset I_2$, ..., $I_n' \subset I_n$ such that f maps I_i' onto I_{i+1}' for $1 \leq i \leq n-1$ and also maps I_n' onto I_1. As a consequence, f^n maps I_1' onto I_1. Lemma 1 implies that f^n has a fixed point $x \in I_1'$. By construction, $f^i(x) \in I_i' \subset I_i$ for $0 \leq i \leq n-1$.

Proposition 5 If the map f has a periodic point of prime period 3, then it has periodic points of any prime period.

Proof: Suppose x_1, x_2, x_3 are points forming a periodic orbit of f, ordered so that $x_1 < x_2 < x_3$. We have that either $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$, or else $f(x_1) = x_3$, $f(x_2) = x_1$, $f(x_3) = x_2$. In the first case, let $I_1 = [x_2, x_3]$ and $I_2 = [x_1, x_2]$. Otherwise we let $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. Then $\circlearrowright I_1 \rightleftarrows I_2$, i.e., $I_1 \to I_2 \to I_1$ and $I_1 \to I_1$.

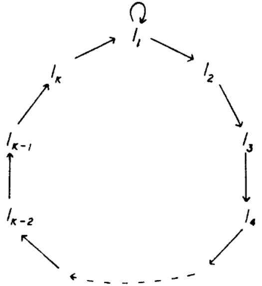
The map f has a periodic point of prime period 3. By Lemma 2, it also has a fixed point. To find a periodic point of prime period n, where $n = 2$ or $n \geq 4$, we notice that

$$I_2 \to \underbrace{I_1 \to I_1 \to \cdots \to I_1}_{n-1 \text{ times}} \to I_2.$$

By Lemma 4, there exists $x \in I_2$ such that $f^n(x) = x$ and $f^i(x) \in I_1$ for $1 \leq i \leq n-1$. If $x \notin I_1$, we obtain that n is the prime period of x. Otherwise $x = x_2$, which leads to a contradiction.

Proposition 6 If the map f has a periodic point of odd prime period $n \geq 5$, then it has a periodic point of any prime period $m \lhd n$.

Proof: It is no loss to assume that f has no periodic points of odd prime periods p, $1 < p < n$. Let $x_1, x_2, \ldots, x_n$ be points of a periodic orbit of prime period n, $x_1 < x_2 < \cdots < x_n$. First we show that one can choose $k \geq 2$ distinct intervals $I_1, I_2, \ldots I_k$ among $[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ that satisfy



Then we show that, in fact, $k = n - 1$.

First we show that one can choose $k \geq 2$ distinct intervals $I_1, I_2, \ldots I_k$ among $[x_1, x_2]$, $[x_2, x_3], \ldots, [x_{n-1}, x_n]$ that satisfy $I_1 \to I_2 \to \cdots \to I_k \to I_1$ and $I_1 \to I_1$.

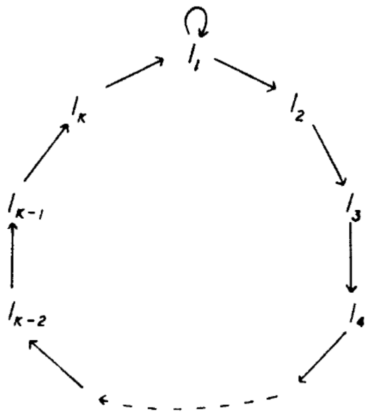
Let $I_1 = [x_j, x_{j+1}]$, where j is the largest index satisfying $f(x_j) > x_j$. Then $f(x_j) \geq x_{j+1}$ and $f(x_{j+1}) \leq x_j$, which implies that $I_1 \to I_1$.

Further, there is an interval $I_\infty = [x_i, x_{i+1}] \neq I_1$ such that $f(x_i)$ and $f(x_{i+1})$ are on different sides of I_1 so that $I_\infty \to I_1$. Indeed, otherwise f would move each x_i to the other side of I_1, which is impossible since n is odd.

Next there are intervals $I_2, \ldots, I_k$ of the form $[x_\ell, x_{\ell+1}]$ such that $I_1, I_2, \ldots, I_k$ are distinct and $I_1 \to I_2 \to \cdots \to I_k = I_\infty$.

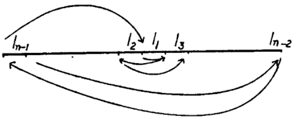
Clearly, $k \leq n-1$. In fact, $k = n-1$ as otherwise we would get a periodic orbit of prime period $n-2$ from the chain

$$I_k \to \underbrace{I_1 \to I_1 \to \cdots \to I_1}_{n-k-1 \text{ times}} \to I_2 \to I_3 \to \cdots \to I_k.$$

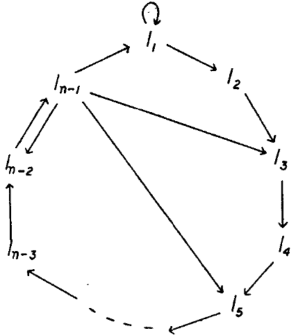


For any diagram of this kind, $k = n - 1$.

As a consequence, $I_s \not\to I_t$ if $t > s+1$ and $I_s \not\to I_1$ if $1 < s < n-1$. It follows that, up to the mirror image, there is only one possible ordering of the intervals $I_1, I_2, \ldots, I_{n-1}$:



This leads to a more refined diagram of coverings:



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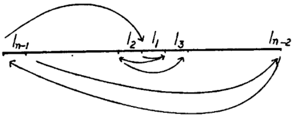


This leads to a more refined diagram of coverings:
$I_1 \to I_1$, $I_1 \to I_2 \to \cdots \to I_{n-1} \to I_1$, and $I_{n-1} \to I_{n-2s}$.

We use this diagram and Lemma 4 to obtain a periodic orbit of f of prime period m for every natural number $m \lhd n$. Namely, in the case $m \geq n-1$ we use a chain

$$I_{n-1} \to \underbrace{I_1 \to I_1 \to \cdots \to I_1}_{m-n+2 \text{ times}} \to I_2 \to I_3 \to \cdots \to I_{n-1}.$$

In the case $1 < m < n-1$, the number m is even, $m = 2s$, and we use a chain $I_{n-1} \to I_{n-2s} \to I_{n-2s+1} \to \cdots \to I_{n-1}$.

Finally, in the case $m = 1$, we use the chain $I_1 \to I_1$.

Lemma 7 $2n \triangleright 2m$ if and only if $n \triangleright m$ for all $n, m \in \mathbb{N}$.

Lemma 8 If x is a periodic point of the map f of prime period n, then x is also a periodic point of f^k of prime period $n/\gcd(n,k)$.

Lemma 9 Assume that for some $n, m > 1$, period n implies period m. Then period $2n$ implies period $2m$.

Proof: Suppose x is a periodic point of the map f of prime period $2n$. Then x is a periodic point of f^2 of prime period n. By assumption, f^2 also has a periodic point y of prime period m. Then $f^{2m}(y) = (f^2)^m(y) = y$ so that y is a periodic point of f of prime period ℓ, where ℓ divides $2m$. By Lemma 8, $\ell = 2m$ if ℓ is even and $\ell = m$ if ℓ is odd. In the former case, we are done. In the latter case, we apply Proposition 5 or 6.

Lemma 10 If f has a periodic point of prime period 4, then it also has a periodic point of prime period 2.

On the converse of Sharkovskii's Theorem

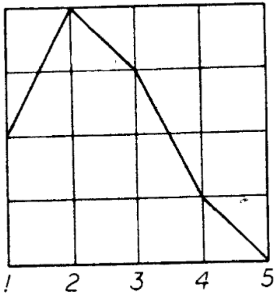
Let $n \in \mathbb{N}$. Consider an arbitrary permutation π of $\{1, 2, \ldots, n\}$ that consists of a single cycle of length n.

We can extend π to a continuous function $f : [1, n] \to [1, n]$ so that f be linear on each of the intervals $[1, 2]$, $[2, 3], \ldots, [n - 1, n]$. Further, we can extend f to a continuous function $f : \mathbb{R} \to \mathbb{R}$ so that f be constant on $(-\infty, 1]$ and on $[n, \infty)$. Then all periodic points of f are in $[1, n]$.

By construction, f has a periodic point of prime period n. One can try to pick π so that there are no periodic points of prime periods $m \triangleright n$.

Period 5 orbit, but no period 3 orbit

Example. $n = 5, \;\; \pi = (1\,3\,4\,2\,5).$



We obtain that $f^3([1,2]) = [2,5]$, $f^3([2,3]) = [3,5]$, $f^3([3,4]) = [1,5]$, $f^3([4,5]) = [1,4]$. Moreover, f^3 is strictly decreasing on $[3,4]$. Therefore f^3 has a unique fixed point, which is also a fixed point of f.

Period 5 orbit, but no period 3 orbit

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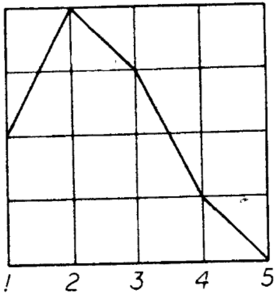

Lemma The map f is expansive on the interval $[1, 5]$.

Theorem The map f is chaotic on the interval $[1, 5]$.