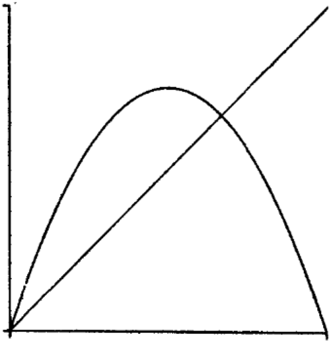
MATH 614

Dynamical Systems and Chaos

Lecture 14: Orbit diagram for the logistic map. Topological Markov chains.

Logistic map

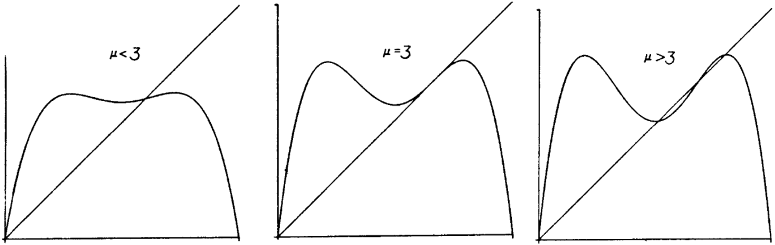


Logistic map $F_\mu(x) = \mu x(1-x)$

Period doubling: logistic map

Logistic map $F_\mu(x) = \mu x(1-x)$, $\mu \approx 3$, $x \approx 2/3$.

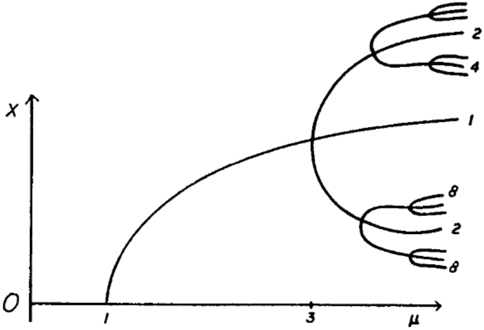
Consider graphs of F_μ^2 for $\mu \approx 3$:



For $\mu < 3$, the fixed point $p_\mu = 1 - \mu^{-1}$ is attracting. At $\mu = 3$, it is not hyperbolic. For $\mu > 3$, the fixed point p_μ is repelling and there is also an attracting periodic orbit of period 2.

Period-doubling route to chaos

The logistic map F_μ has the period doubling bifurcation when the parameter μ passes 3. As μ increases beyond 3, the map undergoes repeated period doublings, namely, the period doubling bifurcation for F_μ^2, then for F_μ^4, then for F_μ^8, and so on.



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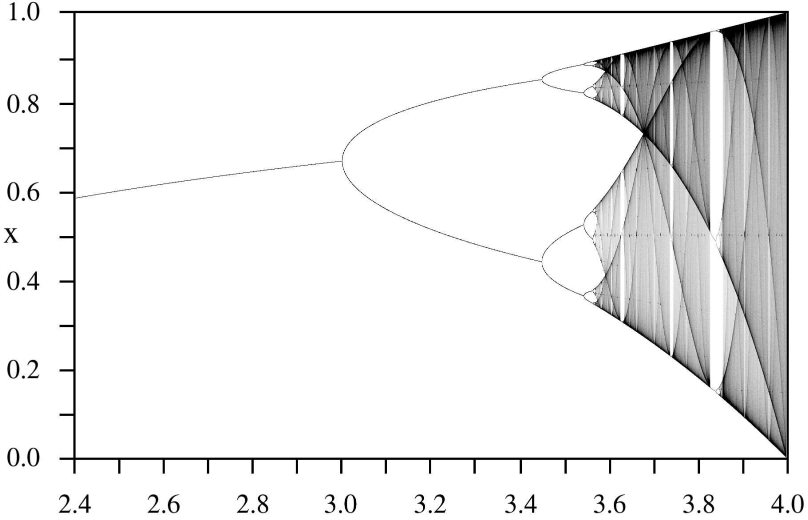
However the period doubling regime ends before μ reaches 4 when the hard chaos develops. To get more information about various kinds of bifurcations for the logistic map, we create the **orbit diagram** as follows. For many equally spaced values of μ, we compute the first 500 points of the orbit of $1/2$, then plot the last 400 of them on the (λ, x)-plane. It is known that the map F_μ has at most one attracting periodic orbit and that the orbit of $1/2$ is always attracted to it.

Orbit diagram for the logistic map



Feigenbaum's universality

For any integer $n \geq 1$ let μ_n be the smallest value of the parameter μ such that the logistic map $F_\mu(x) = \mu x(1-x)$ admits a periodic orbit of prime period n for all $\mu > \mu_n$.

The period-doubling bifurcations occur at $\mu = \mu_2, \mu_4, \mu_8, \ldots$ The limit $\mu_\infty = \lim_{k\to\infty} \mu_{2^k}$ is the smallest value of the parameter μ at which the logistic map starts showing signs of chaotic behaviour.

There exists a limit

$$\lim_{i\to\infty} \frac{\mu_{2^i} - \mu_{2^{i-1}}}{\mu_{2^{i+1}} - \mu_{2^i}} = \delta \approx 4.6692$$

called the **Feigenbaum constant**.

Feigenbaum's universality

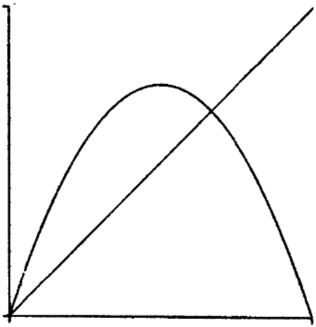
Suppose n is an integer such that $n > 1$ and n is not a power of 2. For all $\mu > \mu_n$ close enough to μ_n, the periodic orbit of prime period n is attracting. As the value of μ increases, this orbit goes through a series of period doublings that occur at some values $\mu = \mu_{n,2}, \mu_{n,4}, \mu_{n,8}, \ldots$

Moreover, the limit

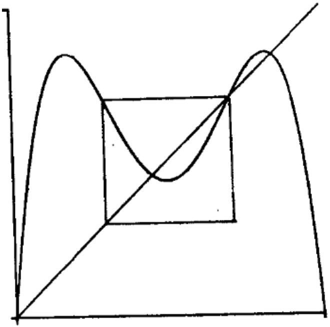
$$\lim_{i\to\infty} \frac{\mu_{n,2^i} - \mu_{n,2^{i-1}}}{\mu_{n,2^{i+1}} - \mu_{n,2^i}}$$

exists and it is the same constant δ as above.

Renormalization



Graph of F_μ



Graph of F_μ^2

Subshift

Given a finite set $\mathcal{A}$ (an alphabet), we denote by $\Sigma_{\mathcal{A}}$ the set of all infinite words over $\mathcal{A}$, i.e., infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$. The **shift** transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$.

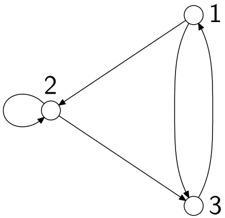
Suppose Σ' is a closed subset of the space $\Sigma_{\mathcal{A}}$ invariant under the shift σ, i.e., $\sigma(\Sigma') \subset \Sigma'$. The restriction of the shift σ to the set Σ' is called a **subshift**.

Suppose W is a collection of finite words in the alphabet $\mathcal{A}$. Let Σ' be the set of all $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that do not contain any element of W as a subword. Then Σ' is a closed, shift-invariant set. Any subshift can be defined this way.

In the case the set W of "forbidden" words can be chosen finite, the subshift is called a **subshift of finite type**. If, additionally, all forbidden words are of length 2, then the subshift is called a **topological Markov chain**.

Topological Markov chains

A topological Markov chain can be defined by a directed graph with the vertex set $\mathcal{A}$ where edges correspond to allowed words of length 2.

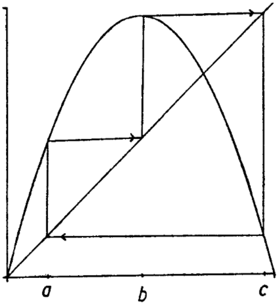


$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

To any topological Markov chain we associate a matrix $M = (m_{ij})$ whose rows and columns are indexed by $\mathcal{A}$ and $m_{ij} = 1$ or 0 if the word ij is allowed (resp., forbidden). The matrix is actually the incidence matrix of the above graph.

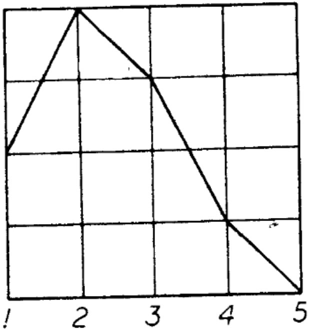
Theorem If for some $n \geq 1$ all entries of the matrix M^n are positive, then the topological Markov chain is chaotic.

Example



For some value of μ, the point $1/2$ is a periodic point of period 3 for the logistic map F_μ. Let $I_0 = [a, b]$ and $I_1 = [b, c]$. The covering diagram of the intervals I_i gives rise to a topological Markov chain over the alphabet $\{0, 1\}$.

Example



Let $I_1 = [1,2]$, $I_2 = [2,3]$, $I_3 = [3,4]$, and $I_4 = [4,5]$. The covering diagram of the intervals I_i gives rise to a topological Markov chain over the alphabet $\{1,2,3,4\}$. Any admissible infinite word is realized as the itinerary of some point $x \in [1,5]$.

Subshifts of finite type

Theorem Any subshift of finite type is topologically conjugate to a topological Markov chain.

Example. $\mathcal{A} = \{0, 1\}$, $W = \{00, 111\}$.

Let us introduce a new alphabet

$$\mathcal{A}' = \{[00], [01], [10], [11]\}$$

and an encoding $\pi : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}'}$ given by

$$\pi(s_1 s_2 s_3 \dots) = ([s_1 s_2][s_2 s_3][s_3 s_4] \dots).$$

For any subshift of finite type over $\mathcal{A}$ with forbidden words of length at most 3, this encoding provides a topological conjugacy with a topological Markov chain over $\mathcal{A}'$.