MATH 614

Dynamical Systems and Chaos

Lecture 20: Stable and unstable sets. Hyperbolic toral automorphisms.

Stable and unstable sets

Let $f : X \to X$ be a continuous map of a metric space (X, d).

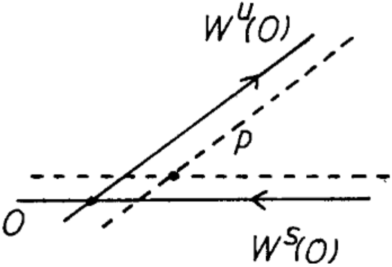
Definition. Two points $x, y \in X$ are **forward asymptotic** with respect to f if $d\big(f^n(x), f^n(y)\big) \to 0$ as $n \to \infty$.
The **stable set** of a point $x \in X$, denoted $W^s(x)$, is the set of all points forward asymptotic to x.

Being forward asymptotic is an equivalence relation on X. The stable sets are equivalence classes of this relation. In particular, these sets form a partition of X.

In the case f is a homeomorphism, we say that two points $x, y \in X$ are **backward asymptotic** with respect to f if $d\big(f^{-n}(x), f^{-n}(y)\big) \to 0$ as $n \to \infty$. The **unstable set** of a point $x \in X$, denoted $W^u(x)$, is the set of all points backward asymptotic to x. The unstable set $W^u(x)$ coincides with the stable set of x relative to the inverse map f^{-1}.

Examples

- Linear map $L : \mathbb{R}^n \to \mathbb{R}^n$.



The stable and unstable sets of the origin, $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$, are transversal subspaces of the vector space $\mathbb{R}^n$. For any point $\mathbf{p} \in \mathbb{R}^n$, the stable and unstable sets are obtained from $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$ by a translation: $W^s(\mathbf{p}) = \mathbf{p} + W^s(\mathbf{0})$, $W^u(\mathbf{p}) = \mathbf{p} + W^u(\mathbf{0})$.

Examples

- One-sided shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$.

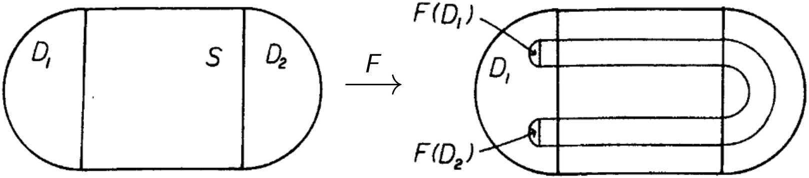
Two infinite words $\mathbf{s} = (s_1 s_2 \dots)$ and $\mathbf{t} = (t_1 t_2 \dots)$ are forward asymptotic if they eventually coincide: $s_n = t_n$ for large n, in which case the orbits of $\mathbf{s}$ and $\mathbf{t}$ under the shift eventually coincide as well. The stable set $W^s(\mathbf{s})$ consists of all infinite words that differ from $\mathbf{s}$ in only finitely many letters. The set $W^s(\mathbf{s})$ is countable and dense in $\Sigma_{\mathcal{A}}$.

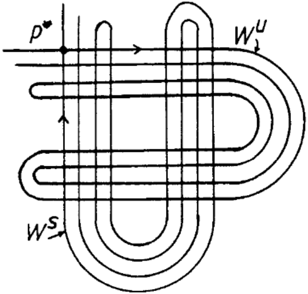
- Two-sided shift $\sigma : \Sigma^{\pm}_{\mathcal{A}} \to \Sigma^{\pm}_{\mathcal{A}}$.

Two bi-infinite words $\mathbf{s} = (\dots s_{-2} s_{-1} . s_0 s_1 s_2 \dots)$ and $\mathbf{t} = (\dots t_{-2} t_{-1} . t_0 t_1 t_2 \dots)$ are forward asymptotic if there exists $n_0 \in \mathbb{Z}$ such that $s_n = t_n$ for $n \geq n_0$. They are backward asymptotic if there exists $n_0 \in \mathbb{Z}$ such that $s_n = t_n$ for $n \leq n_0$.

Examples

- The horseshoe map $F : D \to D$.





Hyperbolic toral automorphisms

Suppose A is an $n \times n$ matrix with integer entries. Let L_A denote a toral endomorphism induced by the linear map $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$. The map L_A is a **toral automorphism** if it is invertible.

Proposition The following conditions are equivalent:

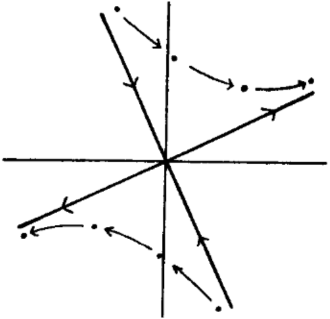
- L_A is a toral automorphism,
- A is invertible and A^{-1} has integer entries,
- $\det A = \pm 1$.

Definition. A toral automorphism L_A is **hyperbolic** if the matrix A has no eigenvalues of absolute value 1.

Theorem Every hyperbolic toral automorphism is chaotic.

Examples of stable and unstable sets

- Hyperbolic toral automorphism $L_A : \mathbb{T}^2 \to \mathbb{T}^2$.



Stable and unstable sets of L_A are images of the corresponding sets of the linear map $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$, under the natural projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$. These sets are dense in the torus $\mathbb{T}^2$.