MATH 614

Dynamical Systems and Chaos

Lecture 23:
Hyperbolic dynamics.

Hyperbolic periodic points

Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map.

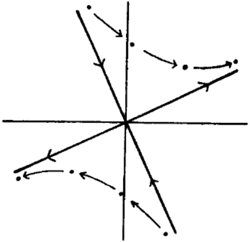
Definition. A fixed point p of the map F is **hyperbolic** if the Jacobian matrix $DF(p)$ has no eigenvalues of absolute value 1 or 0. A periodic point p of period n of the map F is **hyperbolic** if p is a hyperbolic fixed point of the map F^n.

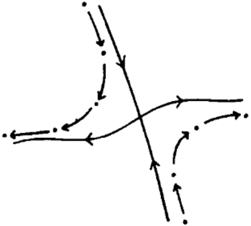
Notice that $DF^n(p) = DF(F^{n-1}(p)) \dots DF(F(p))\, DF(p)$.
It follows that $DF^n(p), DF^n(F(p)), \dots, DF^n(F^{n-1}(p))$ are similar matrices. In particular, they have the same eigenvalues.

Definition. The hyperbolic periodic point p of period n is a **sink** if every eigenvalue λ of $DF^n(p)$ satisfies $0 < |\lambda| < 1$, a **source** if every eigenvalue λ satisfies $|\lambda| > 1$, and a **saddle point** otherwise.

Saddle point

The following figures show the phase portrait of a linear and a nonlinear two-dimensional maps near a saddle point.





Stable and unstable manifolds

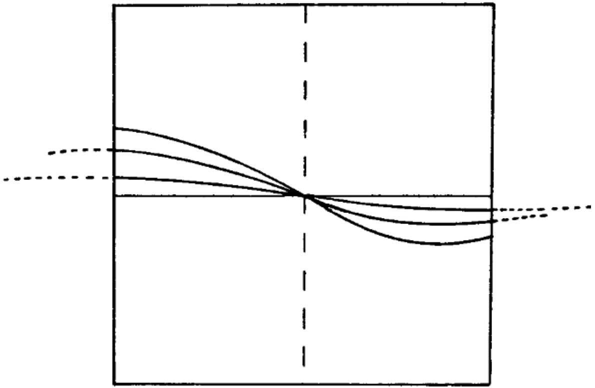
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and suppose p is a saddle point of F of period m.

Theorem There exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ and $\delta > 0$ such that

(i) $\gamma(0) = p$;
(ii) $\gamma'(0)$ is an unstable eigenvector of $DF^m(p)$;
(iii) $F^{-1}(\gamma) \subset \gamma$;
(iv) $\|F^{-n}(\gamma(t)) - F^{-n}(p)\| \to 0$ as $n \to \infty$.
(v) if $\|F^{-n}(x) - F^{-n}(p)\| < \delta$ for all $n \geq 0$, then $x = \gamma(t)$ for some t.

The curve γ is called the **local unstable manifold** of F at p. The **local stable manifold** of F at p is defined as the local unstable manifold of F^{-1} at p.

Stable and unstable manifolds

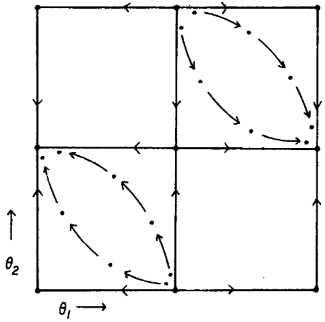


Example

In angular coordinates (θ_1, θ_2) on the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$F\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 - \varepsilon \sin\theta_1\\ \theta_2 + \varepsilon \sin\theta_2\end{pmatrix}.$$

There are 4 fixed points: one source, one sink, and two saddles.

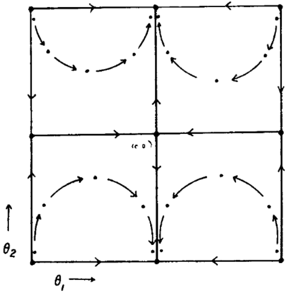


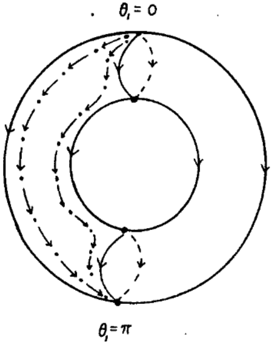
Example

In angular coordinates (θ_1, θ_2) on the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$F\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 + \varepsilon \sin\theta_1\\ \theta_2 + \varepsilon \sin\theta_2 \cos\theta_1\end{pmatrix}.$$

There are 4 fixed points: one source, one sink, and two saddles.





Hyperbolic set

Suppose $F : D \to D$ is a diffeomorphism of a domain $D \subset \mathbb{R}^k$.

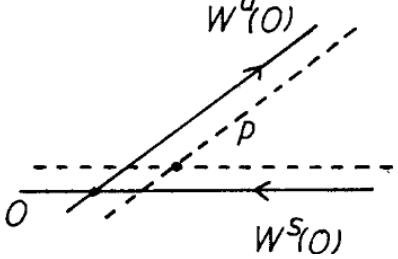
Definition. A set $\Lambda \subset D$ is called a **hyperbolic set** for F if for any $x \in \Lambda$ there exists a pair of subspaces $E^s(x), E^u(x) \subset \mathbb{R}^k$ such that

(i) $\mathbb{R}^k = E^s(x) \oplus E^u(x)$ for all $x \in \Lambda$;

(ii) $DF\big(E^s(x)\big) = E^s(F(x))$ and $DF\big(E^u(x)\big) = E^u(F(x))$ for all $x \in \Lambda$;

(iii) the subspaces $E^s(x)$ and $E^u(x)$ vary continuously with x;

(iv) there is a constant $\lambda > 1$ such that $\|DF(x)\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ for all $\mathbf{v} \in E^u(x)$ and $\|DF(x)\mathbf{v}\| \leq \lambda^{-1}\|\mathbf{v}\|$ for all $\mathbf{v} \in E^s(x)$.


$W^u(O)$
P
O
$W^s(O)$

Hyperbolic set

Conditions (ii) and (iv) imply that
$\|DF^n(x)\mathbf{v}\| \geq \lambda^n \|\mathbf{v}\|$ for all $\mathbf{v} \in E^u(x)$ and
$\|DF^n(x)\mathbf{v}\| \leq \lambda^{-n} \|\mathbf{v}\|$ for all $\mathbf{v} \in E^s(x)$.

Note that condition (iv) may not be preserved under changes of coordinates. We can modify it as follows:

(iv′) there are constants $c, \lambda > 1$ such that
$\|DF^n(x)\mathbf{v}\| \geq c^{-1}\lambda^n \|\mathbf{v}\|$ for all $\mathbf{v} \in E^u(x)$ and
$\|DF^n(x)\mathbf{v}\| \leq c\lambda^{-n} \|\mathbf{v}\|$ for all $\mathbf{v} \in E^s(x)$.

Stable and unstable manifolds

Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a diffeomorphism and suppose Λ is a compact invariant hyperbolic set for F. Assume that $\dim E^u(x) = 1$ for all $x \in \Lambda$ (this is automatic if $k = 2$).

Theorem There exists $\varepsilon > 0$ and, for any $x \in \Lambda$, a smooth curve $\gamma_x : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

(i) $\gamma_x(0) = x$;
(ii) $\gamma_x'(0) \in E^u(x) \setminus \{\mathbf{0}\}$;
(iii) γ_x depends continuously on x;
(iv) $F(\gamma_x) \supset \gamma_{F(x)}$;
(v) $\|F^{-n}(\gamma_x(t)) - F^{-n}(x)\| \to 0$ as $n \to \infty$.

The curve γ_x is called the **local unstable manifold** of F at x. In the case $\dim E^u(x) = d > 1$, the theorem holds as well, with curves γ_x replaced by d-dimensional smooth manifolds. The **local stable manifold** of F at a point x is defined as the local unstable manifold of F^{-1} at x.

Examples of hyperbolic sets

- For any hyperbolic periodic point, the orbit is a hyperbolic set.

- For a hyperbolic toral automorphism, the entire torus is a hyperbolic set. Such a map is called an Anosov map; it is C^1-structurally stable.

- For the horseshoe map, the invariant Cantor set is hyperbolic. It is an example of the so-called Axiom A map (or a Smale map). Such a map is structurally stable.

Shadowing Lemma

Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

Definition. We say that a sequence $x_n, x_{n+1}, \ldots, x_m$ of elements of X is δ-**shadowed** by the orbit of a point $y \in X$ if $d\big(F^i(y), x_i\big) < \delta$ for $n \le i \le m$.
The sequence $x_n, x_{n+1}, \ldots, x_m$ is an ε-**pseudo-orbit** of the map F if $d\big(F(x_{i-1}), x_i\big) < \varepsilon$ for $n < i \le m$.

Theorem (Bowen) Suppose F is a diffeomorphism that admits an invariant hyperbolic set Λ. Then for any $\delta > 0$ there exists $\varepsilon > 0$ such that every ε-pseudo-orbit $x_n, x_{n+1}, \ldots, x_m$ of elements of Λ is δ-shadowed by the orbit of some $y \in \Lambda$.