MATH 614

Dynamical Systems and Chaos

Lecture 28:
Periodic points of holomorphic maps.
Möbius transformations.

Classification of periodic points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a holomorphic function. Suppose that $F(z_0) = z_0$ for some $z_0 \in U$. The fixed point z_0 is called

- attracting if $|F'(z_0)| < 1$;
- repelling if $|F'(z_0)| > 1$;
- neutral if $|F'(z_0)| = 1$.

Now suppose that $F^n(z_1) = z_1$ for some $z_1 \in U$ and an integer $n \geq 1$. The periodic point z_1 is called

- attracting if $|(F^n)'(z_1)| < 1$;
- repelling if $|(F^n)'(z_1)| > 1$;
- neutral if $|(F^n)'(z_1)| = 1$.

The multiplier $(F^n)'(z_1)$ is the same for all points in the orbit of z_1. In particular, all these points are of the same type as z_1. Note that the multiplier is preserved under any holomorphic change of coordinates.

Examples

- $Q(z) = z^2$.

The squaring function has 2 fixed points: 0 and 1.

0 is super attracting: $Q'(0) = 0$.
1 is repelling: $Q'(1) = 2$.

All nonzero periodic points of Q are located on the unit circle. They are all repelling since $|Q'(z)| = 2$ for any z, $|z| = 1$.

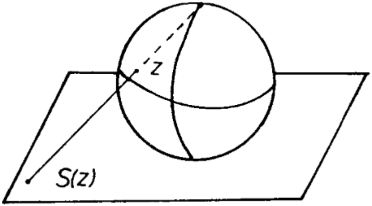
- Any polynomial P of degree at least 2.

Theorem P has infinitely many periodic points. Only finitely many of them can be attracting.

Note that any periodic point z_0 of period n is a root of the polynomial $F(z) = P^n(z) - z$, which has degree $(\deg P)^n$. The root is multiple if $(P^n)'(z_0) = 1$.

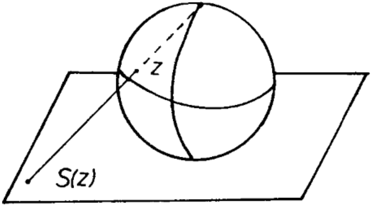
Stereographic projection

Suppose Σ is a sphere in $\mathbb{R}^3$ and Π is the tangent plane at some point $P_s \in \Sigma$. Let P_n be the point of Σ opposite to P_s. Then any straight line through P_n not parallel to Π intersects the plane Π and also intersects the sphere Σ at a point different from P_n.



This gives rise to a map $S : \Sigma \setminus \{P_n\} \to \Pi$, which is a homeomorphism. The map S is referred to as the **stereographic projection**. Note that S maps any circle on Σ onto a circle or a straight line in the plane Π.

The Riemann sphere



Introducing Cartesian coordinates on the plane Π with the origin at P_s, we can identify Π with the complex plane $\mathbb{C}$. The **extended complex plane** $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is $\mathbb{C}$ with one extra point "at infinity". We extend the stereographic projection S to a map $S : \Sigma \to \overline{\mathbb{C}}$ by letting $S(P_n) = \infty$. The topology on $\overline{\mathbb{C}}$ is defined so that S be a homeomorphism.

A holomorphic structure on $\mathbb{C}$ is extended to $\overline{\mathbb{C}}$ by requiring that the map $H : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by $H(z) = 1/z$ for $z \in \mathbb{C} \setminus \{0\}$, $H(0) = \infty$, and $H(\infty) = 0$ be holomorphic.

Complex projective line

The Riemann sphere can also be regarded as the **complex projective line** $\mathbb{CP}^1$. Formally, elements of $\mathbb{CP}^1$ are one-dimensional subspaces of the complex vector space $\mathbb{C}^2$. They are given by their **homogeneous coordinates** $[z : w]$. The complex plane $\mathbb{C}$ is embedded into $\mathbb{CP}^1$ via the map $z \mapsto [z : 1]$. The element $[1 : 0]$ is the point at infinity.

A **projective transformation** T of $\mathbb{CP}^1$ is given by $[z' : w'] = T([z : w])$, where

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = r \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

for a variable $r \neq 0$ and fixed $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \beta\gamma \neq 0$ (so that the matrix is invertible). The matrix is unique up to scaling. In terms of the complex variable z, the map T is given by $T(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$.

Möbius transformations

Definition. A **Möbius transformation** is a rational map of the form $T(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$, regarded as a transformation of the Riemann sphere $\overline{\mathbb{C}}$.

Properties of the Möbius transformations:

- The Möbius transformations form a transformation group.
- Any Möbius transformation is a homeomorphism of $\overline{\mathbb{C}}$.
- Any Möbius transformation is holomorphic.
- Complex affine functions $T(z) = \alpha z + \beta$, $\alpha \neq 0$ are Möbius transformations that fix ∞.
- Complex linear functions $T(z) = \alpha z$, $\alpha \neq 0$ are Möbius transformations that fix 0 and ∞.

More properties of Möbius transformations

- The group of Möbius transformations is generated by linear functions $z \mapsto \alpha z$, translations $z \mapsto z + \beta$, and the map $z \mapsto 1/z$.

- Any Möbius transformation maps circles on the Riemann sphere (which are circles or straight lines in $\mathbb{C}$) onto other circles.

- For any triples z_1, z_2, z_3 and w_1, w_2, w_3 of distinct points on $\overline{\mathbb{C}}$ there exists a unique Möbius transformation T such that $T(z_i) = w_i$, $1 \le i \le 3$.

In the case neither of the six points is ∞, the map is given by

$$\frac{T(z) - w_1}{w_2 - w_1} : \frac{T(z) - w_3}{w_2 - w_3} = \frac{z - z_1}{z_2 - z_1} : \frac{z - z_3}{z_2 - z_3}.$$

More properties of Möbius transformations

- Any Möbius transformation is conjugate (by another Möbius transformation) to a translation or a linear map.

- Any Möbius transformation different from the identity has one or two fixed points.

- If a Möbius transformation has only one fixed point, then this fixed point is neutral.

- If a Möbius transformation has two fixed points, then either both are neutral, or one is attracting while the other is repelling.