MATH 614

Dynamical Systems and Chaos

Lecture 31: The quadratic maps. The Mandelbrot set.

The Julia set

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Informally, the Julia set of P is the set of points where iterates of P exhibit sensitive dependence on initial conditions (chaotic behavior). The Fatou set of P is the set of points where iterates of P exhibit regular, stable behavior.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P.

Quadratic family

The quadratic family $Q_c : \mathbb{C} \to \mathbb{C}$, $c \in \mathbb{C}$,
$Q_c(z) = z^2 + c$.

Examples. • $Q_0 : \mathbb{C} \to \mathbb{C}$, $Q_0(z) = z^2$.
$J(Q_0) = \{z \in \mathbb{C} : |z| = 1\}$.

• $Q_{-2} : \mathbb{C} \to \mathbb{C}$, $Q_{-2}(z) = z^2 - 2$.
$J(Q_{-2}) = [-2, 2]$.

Quadratic family

The quadratic family $Q_c : \mathbb{C} \to \mathbb{C}$, $c \in \mathbb{C}$,
$Q_c(z) = z^2 + c$.

The dynamics of Q_c depends on properties of the post-critical orbit $0, Q_c(0), Q_c^2(0), \ldots$.

Theorem (Fundamental Dichotomy)
For any $c \in \mathbb{C}$,

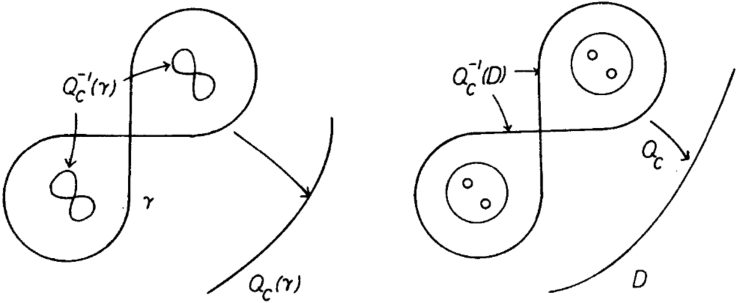
either the post-critical orbit $0, Q_c(0), Q_c^2(0), \ldots$ escapes to ∞, in which case the Julia set $J(Q_c)$ is a Cantor set,

or the post-critical orbit is bounded, in which case the Julia set $J(Q_c)$ is connected.

Lemma If $|z| > \max(|c|, 2)$ then $|Q_c^n(z)| \to \infty$ as $n \to \infty$.

Suppose γ_0 is a simple closed curve in $\mathbb{C}$ and D_0 is the domain bounded by γ_0. Let $\gamma_1 = Q_c^{-1}(\gamma_0)$ and $D_1 = Q_c^{-1}(D_0)$. Then γ_1 is the boundary of D_1.

If $c \in D_0$ then γ_1 is a simple closed curve. If $c \in \gamma_0$ then γ_1 is a "figure eight". Otherwise γ_1 consists of two curves.



If D_0 is a large disk then $D_1 = Q_c^{-1}(D_0)$ is contained in D_0. Hence $D_0 \supset D_1 \supset D_2 \supset \dots$, where $D_{n+1} = Q_c^{-1}(D_n)$, $n \geq 0$.

Note that the set $K(Q_c) = \bigcap_{n=0}^{\infty} D_n$ does not depend on the choice of D_0.

Now either $c \in D_n$ for all n, in which case each D_n is a simply connected domain so that $K(Q_c)$ is also connected. In this case, $J(Q_c)$ is the boundary of $K(Q_c)$.

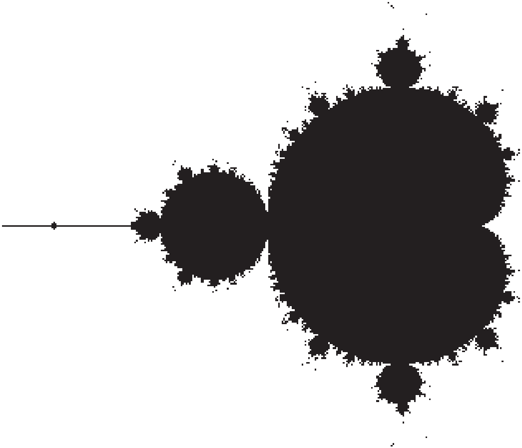
Otherwise $c \in D_n \setminus D_{n+1}$ for some n, in which case D_{n+k} has 2^{k-1} connected components, $k = 0, 1, 2, \dots,$ so that $K(Q_c)$ is a Cantor set. In this case, $J(Q_c) = K(Q_c)$.

The Mandelbrot set

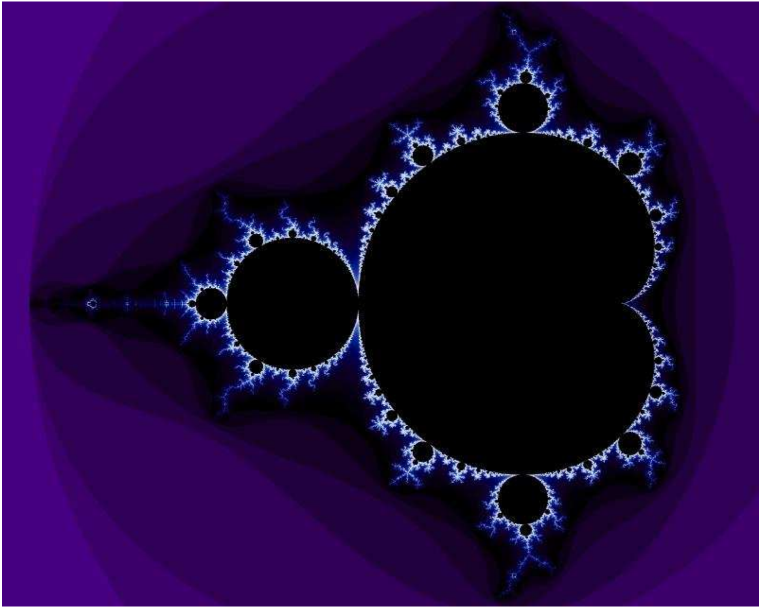
Definition. The **Mandelbrot set** $\mathcal{M}$ is the set of all $c \in \mathbb{C}$ such that $|Q_c^n(0)| \not\to \infty$ as $n \to \infty$.

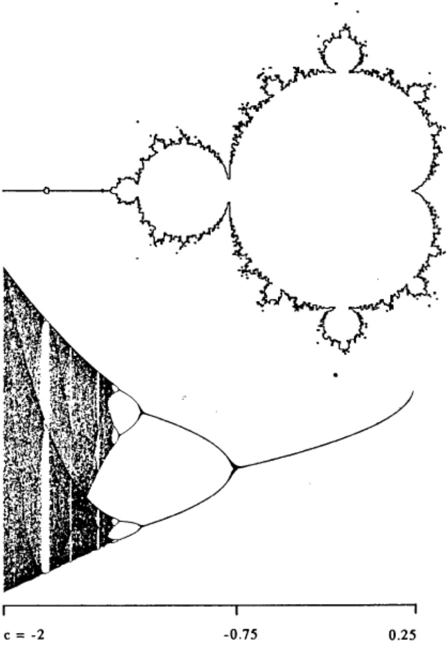
$c \in \mathcal{M}$ if and only if the Julia set $J(Q_c)$ is connected.

The Mandelbrot set $\mathcal{M}$ is the bifurcation diagram for the quadratic family.

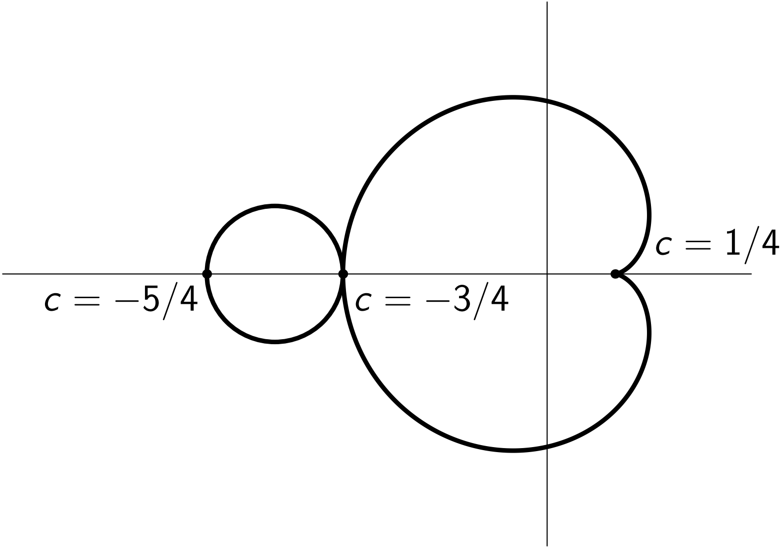


The Mandelbrot set

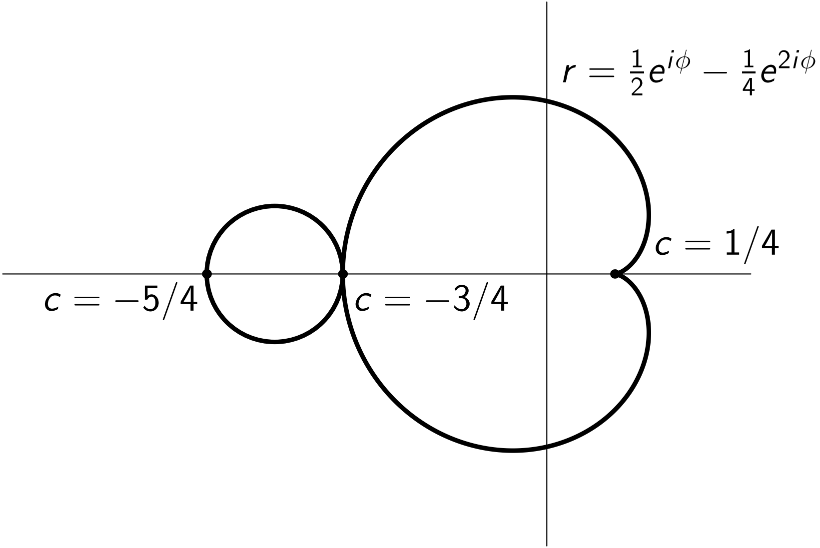




c = -2
-0.75
0.25



The main cardioid and the period 2 bulb

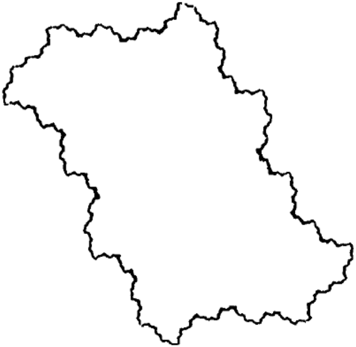


The main cardioid and the period 2 bulb

Theorem 1 For any c within the main cardioid, the Julia set $J(Q_c)$ is a simple closed curve.

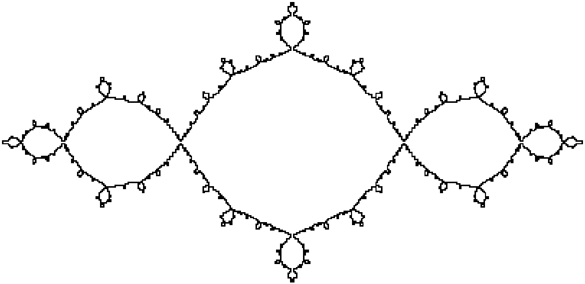
The region bounded by this curve is the basin of attraction of an attracting fixed point.

If $c \notin \mathbb{R}$ then $J(Q_c)$ contains no smooth arc.



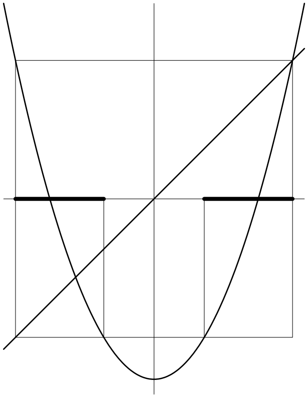
$J(Q_c),\ c = i/2$

Theorem 2 For any c within the period 2 bulb, $J(Q_c)$ is the closure of countably many simple closed curves. The interior of $K(Q_c)$ has countably many connected components and coincides with the basin of attraction of an attracting periodic orbit of period 2.



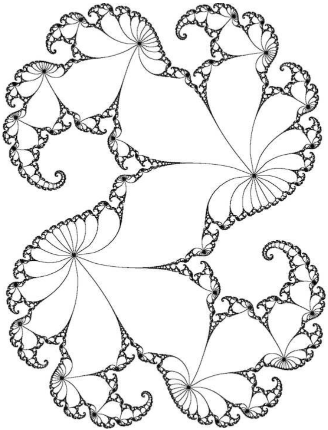
$J(Q_c),\ c = -1$

Theorem 3 For any $c \notin \mathcal{M}$, the Julia set $J(Q_c)$ is a Cantor set and the restriction of Q_c to $J(Q_c)$ is conjugate to the one-sided shift on two letters.



Q_c, $c < -2$. $J(Q_c) = K(Q_c)$ is a Cantor set.

Attracting petals



The Julia set $J(Q_c)$, where $c \approx 0.29 + 0.176i$.

c is chosen on the boundary of the main cardioid so that Q_c has a fixed point with multiplier $\exp(\frac{2\pi i}{15})$.