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On the uniform distribution of orbits of finitely generated groups and semigroups of plane isometries

Ya. B. Vorobets

Abstract. Actions by isometries on a Euclidean plane of free groups and free semigroups with arbitrarily many generators and of free products of groups of order 2 are considered. It is shown that, in the typical case, all orbits of the action are uniformly distributed in the plane. The actions for which the distribution of orbits fails to be uniform are explicitly described.

Bibliography: 5 titles.

§ 1. Introduction

Let P be an affine Euclidean plane. We denote by $\mathcal{G}$ the group of isometries of the plane P, that is, of transformations preserving the distances between points. Each element of the group $\mathcal{G}$ is an affine transformation. We denote by $\mathcal{G}_+$ the group of orientation-preserving isometries of the plane P and by $\mathcal{G}_0$ the group of parallel translations. The rotations of the plane form the set $\mathcal{G}_+ \setminus \mathcal{G}_0$, and the axial and translational symmetries form the set $\mathcal{G} \setminus \mathcal{G}_+$. The group $\mathcal{G}_+$ is an index-two subgroup of $\mathcal{G}$. The group $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$ and $\mathcal{G}_+$.

Let G be a countable group or semigroup. By an *action* of the (semi)group G on the plane P by isometries we mean a (semi)group homomorphism $d\colon G \to \mathcal{G}$. The *orbit* $O_d(x)$ of a point $x \in P$ under the action d is a sequence $\{d(g)x\}_{g\in G}$ of points in the plane indexed by elements of the (semi)group G. By an *orbit of the action* d we mean an orbit of some point $x \in P$ under d.

In the present paper we assume that the group or semigroup G is finitely generated and, moreover, equipped with a fixed set of generators $a_1, \ldots, a_k$. This enables one to introduce a length function on G. By the *length* $|g|$ of an arbitrary element $g \in G$ we mean the smallest integer m such that g can be expanded in a product $g_1 g_2 \cdots g_m$ of factors belonging to the sequence $a_1, \ldots, a_k$ if G is a semigroup or to the sequence a_1, a_1^{-1}, $\ldots$, a_k, a_k^{-1} if G is a group. The length of the identity element is set to be equal to zero. The length function equips the (semi)group G with a partial ordering which works on each orbit of the action d of this (semi)group. We use this ordering to define the concept of uniformly distributed orbit. Let x be a point and let E be a subset of the plane P. For a positive integer n we denote by $N_{d,x}^{(n)}(E)$ the ratio of the number of elements $g \in G$ of length n for which

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$d(g)x \in E$ to the number of all elements of length n in G. An orbit $O_d(x)$ is said to be *uniformly distributed* in the plane if

$$\lim_{n\to\infty} \frac{N_{d,x}^{(n)}(E_1)}{N_{d,x}^{(n)}(E_2)} = \frac{\mu(E_1)}{\mu(E_2)}$$

for each pair E_1, E_2 of Jordan measurable subsets of positive measure of the plane, where μ is Lebesgue measure on P.

Since the elements $a_1, \dots, a_k$ generate the (semi)group G, it follows that the action d is uniquely determined by the isometries $A_1 = d(a_1), \dots, A_k = d(a_k)$, and we denote this action by $G[A_1, \dots, A_k]$. In what follows, for G we shall take the free semigroup FSG_k and the free group FG_k with k generators, and also the free product of k groups of order 2, $\mathbb{Z}_2^{*k} = \langle a_1, \dots, a_k \mid a_1^2 = \dots = a_k^2 = 1\rangle$. We note that the actions of $\mathrm{FG}_k[A_1, \dots, A_k]$ and $\mathrm{FSG}_k[A_1, \dots, A_k]$ are well defined for arbitrary $A_1, \dots, A_k \in \mathcal{G}$. The action $\mathbb{Z}_2^{*k}[A_1, \dots, A_k]$ is well defined if $A_1, \dots, A_k$ are involutions. In the present paper we study asymptotic properties of the means $N_{d,x}^{(n)}(E)$ and, in particular, the problem of the uniform distribution of orbits for actions d of the above-mentioned kinds. The corresponding problem was posed (among others) by Arnol'd and Krylov [1]. The investigations were started by Kazhdan [2] and Guivarc'h [3] and continued by this author in [4] and [5]; the results thus obtained are summarized in Theorems 1.1 and 1.2 stated below.

To formulate these theorems we require another definition. Let Q be a strip (a domain bounded by two parallel lines), a square, or a regular triangle in the plane P. Consider the reflections of the figure Q relative to each of its boundary lines, repeat the same for the reflections of Q, and so on ad infinitum. As a result we obtain a partitioning of the plane P by countably many straight lines into figures congruent to Q. We refer to this partitioning as the *lattice* generated by the figure Q and denote it by $\mathcal{L}_Q$. We say that an isometry $A \in \mathcal{G}$ takes the lattice $\mathcal{L}_Q$ to itself if this isometry takes the lines defining the partitioning $\mathcal{L}_Q$ to one another.

Theorem 1.1 [5]. *Let $A_1, \dots, A_k \in \mathcal{G}$, and let*

$$d = \mathrm{FSG}_{2k}[A_1, A_1^{-1}, \dots, A_k, A_k^{-1}],$$

$d = \mathrm{FG}_k[A_1, \dots, A_k]$, or $d = \mathbb{Z}_2^{*k}[A_1, \dots, A_k]$ (if $A_1, \dots, A_k$ are involutions). If the isometries $A_1, \dots, A_k$ have no common fixed point and preserve no common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle, then all orbits of the action d are uniformly distributed in the plane. Moreover, the estimates

$$I_1 \mu(E) n^{-1} \leqslant N_{d,x}^{(n)}(E) \leqslant I_2 \mu(E) n^{-1},$$

where I_1 and I_2 are positive constants depending only on the action d, hold for sufficiently large values of n for each Jordan-measurable set E of positive measure and each point x.

Theorem 1.2 [4]. *Let $A_1, A_2 \in \mathcal{G}_+$. If the isometries A_1 and A_2 have no common fixed point and preserve no common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle, then all orbits of the action $d = \mathrm{FSG}_2[A_1, A_2]$ are uniformly distributed in the plane.*

The main results of the present paper are the following two theorems. The first of them refines Theorem 1.1 and the second generalizes Theorem 1.2.

Theorem 1.3. *Let $A_1, \dots, A_k \in \mathcal{G}$ and let*

$$d = \mathrm{FSG}_{2k}[A_1, A_1^{-1}, \dots, A_k, A_k^{-1}]$$

or $d = \mathrm{FG}_k[A_1, \dots, A_k]$ (or $d = \mathbb{Z}_2^{*k}[A_1, \dots, A_k]$, provided that $A_1, \dots, A_k$ are involutions). If the isometries $A_1, \dots, A_k$ have no common fixed point and preserve no common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle, then there exists $I_0 > 0$ such that $\lim_{n\to\infty} nN_{d,x}^{(n)}(E) = I_0\mu(E)$ for each Jordan measurable subset E of P and each point $x \in P$. In addition, all orbits of the action d are uniformly distributed in the plane.

Theorem 1.4. *Let $A_1, \dots, A_k \in \mathcal{G}$. If the isometries $A_1, \dots, A_k$ have no common fixed point and preserve no common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle, then the action $d = \mathrm{FSG}_k[A_1, \dots, A_k]$ satisfies one of the following two conditions:*

(1) *there exists $I_0 > 0$ such that $\lim_{n\to\infty} nN_{d,x}^{(n)}(E) = I_0\mu(E)$ for each (Jordan measurable) subset E of P and each point $x \in P$ (in this case all orbits of the action d are uniformly distributed in the plane);*

(2) *there exists $I_0 > 0$ such that for each bounded subset E of P and each point $x \in P$ the sequence $\{\exp(I_0 n)N_{d,x}^{(n)}(E)\}$ is bounded.*

Condition (2) holds only in the following cases:

- (i) *$A_1, \dots, A_k$ are parallel translations by vectors with non-zero sum;*
- (ii) *each isometry $A_1, \dots, A_k$ is either a parallel translation or an axial or a translational symmetry with axis parallel to a fixed line l, and for some (and therefore for each) point $x \in P$ the sum of the vectors $A_j x - x$, $j = 1, \dots, k$, is not orthogonal to the line l.*

§§ 2, 3 of the present paper are devoted to the proof of Theorems 1.3 and 1.4. We now discuss these statements. If the isometries $A_1, \dots, A_k$ have a common fixed point, then each orbit of the action d lies on a circle with centre at this point. If the isometries $A_1, \dots, A_k$ preserve a common lattice generated by a strip, then each orbit lies in the union of countably many parallel lines with pairwise distances bounded below by some $\varepsilon > 0$. If these isometries preserve a common lattice generated by a square or a regular triangle, then the points in each orbit form a discrete subset of the plane. In each case the orbits of the action d are nowhere dense in the plane. On the other hand, if the orbits of the action d are uniformly distributed in the plane P, then these orbits are dense in P. By the general conjecture in [1], *an arbitrary orbit of the action d is uniformly distributed in the plane whenever this orbit is dense.* As we see, Theorem 1.3 (as well as Theorem 1.1) confirms this conjecture for the action of the groups FG_k and $\mathbb{Z}_2^{*k}$, and Theorem 1.4 confirms this conjecture for the actions of the semigroup FSG_k in all cases except (i) and (ii).

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§ 2. Uniform distribution

For an isometry $A \in \mathcal{G}$ we denote by $u[A]$ the linear operator acting on functions in the plane P by the formula $(u[A]f)(x) = f(Ax)$, $x \in P$. The restriction of $u[A]$ to the space $L_2(P,\mu)$ is a unitary operator. We denote by $\mathfrak{U}$ the class of operators that are finite linear combinations of operators of the form $u[A]$. Since $u[A]u[B] = u[BA]$ for $A, B \in \mathcal{G}$, the class $\mathfrak{U}$ is an operator algebra.

We denote by F the Fourier transformation of $\mathbb{R}^2$. We regard F as a unitary automorphism of the space $L_2(\mathbb{R}^2)$. The action of the operators F and F^{-1} on a function $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ is described by the formulae

$$(Ff)(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(\lambda,x)} f(x)\,dx,$$
$$(F^{-1}f)(x) = (Ff)(-x).$$

Let $\mathcal{F}$ be the class of functions $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ whose Fourier image Ff is differentiable and has compact support. We introduce a Cartesian coordinate system ξ in the plane P and treat ξ as an isometric map of P onto the Euclidean space $\mathbb{R}^2$. The ξ-coordinates enable one to regard an arbitrary function f on the plane P as a function on $\mathbb{R}^2$ (by identifying it with the function $f\xi^{-1}$). In particular, one can regard $\mathcal{F}$ as a class of functions on the plane P.

Lemma 2.1 [5]. *The class $\mathcal{F}$ does not depend on the choice of Cartesian coordinates on the plane P and is invariant with respect to operators in the algebra $\mathfrak{U}$. For each Jordan measurable subset E of P and each $\varepsilon > 0$ there exists a pair of functions $f_+, f_- \in \mathcal{F}$ such that $f_- \leqslant \chi_E \leqslant f_+$ in the entire plane and $\int (f_+ - f_-)\,d\mu < \varepsilon$.*

We denote by the symbol S^1 of a circle the quotient space $\mathbb{R}/2\pi\mathbb{Z}$ equipped with the natural topology and Lebesgue measure. An important role in what follows is played by the map $\Phi\colon S^1 \to \mathbb{R}^2$ given by the formula $\Phi(t) = (\cos t, \sin t)$, $t \in \mathbb{R}/2\pi\mathbb{Z}$. For each $R > 0$ the map $R\Phi$ is a homeomorphism of S^1 onto the circle of radius R in $\mathbb{R}^2$ with centre at the origin. Let $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$. Then the Fourier image Ff of f is continuous. For each $R > 0$ we denote by f_R the continuous function on S^1 defined by the formula $f_R(t) = (Ff)(R\Phi(t))$, $t \in S^1$. The function f_R is called the *radial component* of f corresponding to the radius R.

The Cartesian coordinate system ξ enables one to regard isometries in the group $\mathcal{G}$ as transformations of $\mathbb{R}^2$ and, accordingly, to regard operators in the algebra $\mathfrak{U}$ as operators acting on functions on $\mathbb{R}^2$. By the *radial operator* u_R corresponding to $u \in \mathfrak{U}$ and a radius $R > 0$ we mean a bounded linear operator on $L_2(S^1)$ taking the radial component f_R of each function $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ to the radial component $(uf)_R$ of the function uf. We note that the definition of the operator u_R depends on the coordinate system ξ.

Lemma 2.2 [5]. *The radial operator u_R is uniquely defined for each $u \in \mathfrak{U}$ and each $R > 0$. For each $R > 0$ the correspondence $u \mapsto u_R$ is a homomorphism of the algebra $\mathfrak{U}$ into the algebra of linear operators in $L_2(S^1)$. The radial operator $u[A]_R$ is unitary for each $A \in \mathcal{G}$ and each $R > 0$. If A is an isometry preserving the origin, then $u[A]_R h = h\Phi^{-1}A\Phi$ for all $R > 0$ and $h \in L_2(S^1)$. If A is the parallel translation by a vector $v \in \mathbb{R}^2$, then $u[A]_R$ is the operator of multiplication by the function $e^{iR(\Phi,v)}$.*

Let G be a finitely generated group or semigroup with distinguished set of generators and let d be an action of G in the plane P by isometries. We assign to the action d a sequence of *averaging* operators $C^{(0)}$, $C^{(1)}$, $C^{(2)}$, ... belonging to the algebra $\mathfrak{U}$. The operator $C^{(n)}$ is defined by the equality

$$C^{(n)} = m_n^{-1} \sum_{|g|=n} u[d(g)],$$

where m_n is the number of elements of length n in G. If G is a semigroup without an identity element, then the operator $C^{(0)}$ is not defined.

Proposition 2.3. *Let $C_R^{(n)}$, $R > 0$, $n = 0, 1, \dots$, be the radial operators corresponding to the averaging operators $C^{(0)}, C^{(1)}, \dots$ in some coordinate system ξ. Suppose that the operators $C_R^{(n)}$ have the following properties:*

(P1) *there exist positive constants I_1, I_2, and R_0 such that*

$$\|C_R^{(n)}\| \leqslant I_1(1 - I_2R^2)^n, \qquad 0 < R \leqslant R_0;$$

(P1b) *there exist positive constants I_0 and R_0 such that*

$$n \int_0^{R_0} R(C_R^{(n)}1, 1)\, dR \to I_0 \quad \text{as } n \to \infty;$$

(P2) *for all R_1 and R_2, $0 < R_1 < R_2$, there exist constants $I > 0$ and $\rho \in (0, 1)$ such that $\|C_R^{(n)}\| \leqslant I\rho^n$ for $R_1 \leqslant R \leqslant R_2$.*

Then

$$\lim_{n\to\infty} nN_{d,x}^{(n)}(E) = (2\pi)^{-2} I_0 \mu(E)$$

for each Jordan measurable subset E of P and each point $x \in P$. As a consequence, every orbit of the action d is uniformly distributed in the plane.

Proof. Since $|(C_R^{(n)}1, 1)| \leqslant 2\pi\|C_R^{(n)}\|$, it follows from property (P2) that for each $R_1 > 0$ the quantity $n \int_{R_0}^{R_1} R(C_R^{(n)}1, 1)\, dR$ approaches zero as $n \to \infty$. Thus, the property (P1b) persists if one chooses the constant R_0 in an arbitrary way, and the value of the constant I_0 does not depend on this arbitrariness. This enables one to assume that the constant R_0 in conditions (P1) and (P1b) is the same and $I_2R_0^2 < 1$. Then for $0 < R \leqslant R_0$ we have the inequalities $0 < 1 - I_2R^2 \leqslant \exp(-I_2R^2)$; as a consequence, $\|C_R^{(n)}\| \leqslant I_1 \exp(-I_2nR^2)$.

We claim that the validity of properties (P1), (P1b), (P2) is independent of the choice of the coordinate system in which one calculates the radial operators $C_R^{(n)}$, and so is also the value of the constant I_0 in the condition (P1b). Let ξ_1 be a Cartesian coordinate system in the plane P distinct from ξ. Let $\widetilde{C}_R^{(n)}$, $R > 0$, $n = 0, 1, \dots$, be the family of radial operators corresponding to the operators $C^{(0)}, C^{(1)}, \dots$ in the coordinate system ξ_1. We set $A = \xi^{-1}\xi_1$. Obviously, $A \in \mathcal{G}$. One can readily see that the operator $C^{(n)}$ acts on functions on $\mathbb{R}^2$ with respect to the coordinate system ξ_1 in just the same way as the operator $u[A]^{-1}C^{(n)}u[A]$ acts on these functions with respect to the coordinate system ξ. Hence $\widetilde{C}_R^{(n)} = u[A]_R^{-1} C_R^{(n)} u[A]_R$, where the operator $u[A]_R$ is calculated in the coordinate system ξ. Since the operator $u[A]_R$ is unitary, it follows that $\|\widetilde{C}_R^{(n)}\| = \|C_R^{(n)}\|$ and $(\widetilde{C}_R^{(n)}1, 1) = (C_R^{(n)}(u[A]_R 1), u[A]_R 1)$. By the first equality properties (P1) and (P2) do not depend on the choice of the coordinate system, and by the second

$$|(\widetilde{C}_R^{(n)}1, 1) - (C_R^{(n)}1, 1)| \leqslant 4\pi\|C_R^{(n)}\| \sup|u[A]_R 1 - 1|.$$

By Lemma 2.2 we obtain $u[A]_R 1 = e^{iR(\Phi, v)}$, where the vector $v \in \mathbb{R}^2$ depends only on A and ξ. Hence for $0 < R \leqslant R_0$ we have $\sup|u[A]_R 1 - 1| \leqslant I_* R$, where the constant $I_* > 0$ depends only on A, ξ, and R_0. Moreover, $|(\widetilde{C}_R^{(n)}1, 1) - (C_R^{(n)}1, 1)| \leqslant 4\pi I_* R\|C_R^{(n)}\| \leqslant 4\pi I_* I_1 R \exp(-I_2 n R^2)$. As a consequence, we obtain

$$\begin{aligned} n\int_0^{R_0} R|(\widetilde{C}_R^{(n)}1, 1) - (C_R^{(n)}1, 1)|\,dR &\leqslant 4\pi I_* I_1 n \int_0^{R_0} R^2 \exp(-I_2 n R^2)\,dR \\ &= 4\pi I_* I_1 n^{-1/2} \int_0^{n^{1/2}R_0} R^2 \exp(-I_2 R^2)\,dR \\ &\leqslant 4\pi I_* I_1 n^{-1/2} \int_0^{\infty} R^2 \exp(-I_2 R^2)\,dR, \end{aligned}$$

and the right-hand side approaches zero as $n \to \infty$. Thus, the operators $\widetilde{C}_R^{(n)}$, similarly to the $C_R^{(n)}$, have property (P1b) with the same value of the constant I_0.

We now choose a point $x \in P$. In accordance with the above, one can assume without loss of generality that x is the origin of the system ξ. Consider an arbitrary function $f \in \mathfrak{F}$. For a positive integer n we set

$$\alpha_n = (2\pi)^{-2} n \int_0^{R_0} R(C_R^{(n)}1, 1)\,dR,$$
$$\beta_n(f) = n^{3/2}\left((C^{(n)}f)(x) - n^{-1}\alpha_n \int f\,d\mu\right).$$

Then

$$(C^{(n)}f)(x) = n^{-1}\alpha_n \int f\,d\mu + n^{-3/2}\beta_n(f).$$

By Lemma 2.1, $C^{(n)}f \in \mathfrak{F}$. Regarding $C^{(n)}f$ as a function on $\mathbb{R}^2$ (with respect to the ξ-coordinates) we obtain

$$(C^{(n)}f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} (F(C^{(n)}f))(\lambda)\,d\lambda = \frac{1}{2\pi}\int_0^{\infty} R(C_R^{(n)}f_R, 1)\,dR.$$

Bearing in mind that $\int f\,d\mu = 2\pi(Ff)(0)$ and $|(C_R^{(n)}h, 1)| \leqslant 2\pi\|C_R^{(n)}\|\sup|h|$ for each function $h \in L_2(S^1)$, we arrive at the estimate $|\beta_n(f)| \leqslant J_{1,n} + J_{2,n}$, where

$$J_{1,n} = n^{3/2}\int_0^{R_0} R\|C_R^{(n)}\|\sup|f_R - (Ff)(0)|\,dR,$$
$$J_{2,n} = n^{3/2}\int_{R_0}^{\infty} R\|C_R^{(n)}\|\sup|f_R|\,dR.$$

Since $f \in \mathcal{F}$, there exist constants $R_1 > R_0$ and $I_f > 0$ such that $f_R = 0$ for $R \geqslant R_1$ and $\sup|f_R - (Ff)(0)| \leqslant I_f R$ for $0 < R \leqslant R_0$. Since

$$\sup|f_R| \leqslant \sup|Ff| \leqslant \frac{1}{2\pi}\int |f|\,d\mu$$

for each $R > 0$, it follows by property (P2) that $J_{2,n} \to 0$ as $n \to \infty$. Further,

$$J_{1,n} \leqslant n^{3/2}\int_0^{R_0} I_f I_1 R^2 \exp(-I_2 n R^2)\,dR \leqslant I_f I_1 \int_0^{\infty} R^2 \exp(-I_2 R^2)\,dR.$$

The sequence $\{\beta_n(f)\}$ is therefore bounded. It follows by property (P1b) that $\lim_{n\to\infty}\alpha_n = (2\pi)^{-2} I_0 > 0$. Hence

$$n(C^{(n)}f)(x) = \alpha_n \int f\,d\mu + n^{-1/2}\beta_n(f) \to (2\pi)^{-2} I_0 \int f\,d\mu \quad \text{as } n \to \infty.$$

By Lemma 2.1, for each Jordan measurable subset E of P and each $\varepsilon > 0$ there exist functions $f_+, f_- \in \mathcal{F}$ such that $f_- \leqslant \chi_E \leqslant f_+$ in the entire plane,

$$\int f_+\,d\mu < \mu(E) + \varepsilon, \qquad \int f_-\,d\mu > \mu(E) - \varepsilon.$$

Obviously, $(C^{(n)}\chi_E)(x) = N_{d,x}^{(n)}(E)$ for each positive integer n, and therefore

$$(C^{(n)}f_-)(x) \leqslant N_{d,x}^{(n)}(E) \leqslant (C^{(n)}f_+)(x).$$

By the above,

$$\lim_{n\to\infty} n(C^{(n)}f_+)(x) = (2\pi)^{-2} I_0 \int f_+\,d\mu,$$
$$\lim_{n\to\infty} n(C^{(n)}f_-)(x) = (2\pi)^{-2} I_0 \int f_-\,d\mu.$$

As a consequence, all limit points of the sequence $\{(2\pi)^2 I_0^{-1} n N_{d,x}^{(n)}(E)\}$ belong to the interval

$$\left[\int f_-\,d\mu, \int f_+\,d\mu\right] \subset [\mu(E) - \varepsilon, \mu(E) + \varepsilon].$$

Since ε was arbitrarily chosen, it follows that $\lim_{n\to\infty} n N_{d,x}^{(n)}(E) = (2\pi)^{-2} I_0 \mu(E)$. Since the set E was arbitrary, the orbit $O_d(x)$ is uniformly distributed in the plane. Finally, as noted above, the constant I_0 does not depend on the choice of the coordinate system ξ (in particular, on the point x), but depends only on the action d.

Below in this section we shall obtain sufficient conditions for an abstract family $C_R^{(n)}$, $R > 0$, $n = 0, 1, \ldots$, of linear bounded operators on the space $L_2(S^1)$ to have properties (P1), (P1b), and (P2). We shall consider the case $C_R^{(n)} = p_n(C_R)$, where $\{C_R\}_{R>0}$ is a family of bounded operators in the space $L_2(S^1)$ and

$p_0, p_1, \dots$ are polynomials with real coefficients. The following properties of the operators C_R, $R > 0$, are useful for our aims:

(V1) there exist positive constants I_1 and R_0 such that $\|C_R\| \leqslant 1 - I_1 R^2$ for $0 < R \leqslant R_0$;

(V1′) there exist positive constants I_1, R_0, and a positive integer m such that $\|C_R^m\| \leqslant 1 - I_1 R^2$ for $0 < R \leqslant R_0$;

(V1b) there exist constants $I_0 > 1$, $R_0 > 0$, a measurable function g on S^1, and a map $(0, R_0] \ni R \mapsto h_R \in L_2(S^1)$ such that $I_0^{-1} \leqslant g \leqslant I_0$ on the entire circle, $\|h_R - 1\| \leqslant I_0 R$ and $\|C_R h_R - (1 - gR^2) h_R\| \leqslant I_0 R^3$ for $0 < R \leqslant R_0$, and for each $R \in (0, R_0]$ the operator C_R commutes with the operator of multiplication by the function g;

(V2) $\|C_R\| < 1$ for each $R > 0$;

(V2′) for each $R_1 > 0$ there exists a positive integer M such that $\|C_R^M\| < 1$ for $0 < R \leqslant R_1$.

Let $p_0, p_1, p_2, \dots$ be polynomials satisfying the following conditions:

(W1) there exist positive constants I_2, I_3, and ε such that

$$\max(|p_n(1 - z)|, |p_n(-1 + z)|) \leqslant I_2 (1 - I_3 z)^n, \qquad 0 \leqslant z \leqslant \varepsilon;$$

(W1b) there exist positive constants I_4, I_5, I_6, and ε such that

$$|p_n(1 - z) - I_4 (1 - I_5 z)^n| \leqslant I_6 (z + n z^2), \qquad 0 \leqslant z \leqslant \varepsilon;$$

(W2) for each $z_0 \in (0, 1)$ there exist positive constants I and ρ such that $|p_n(z)| \leqslant I \rho^n$ for $-z_0 \leqslant z \leqslant z_0$.

Proposition 2.4. *Let $\{C_R\}_{R>0}$ be a family of bounded self-adjoint operators in the space $L_2(S^1)$ continuously dependent on the parameter R and let $p_0, p_1, \dots$ be a sequence of polynomials with real coefficients. If the family $\{C_R\}_{R>0}$ has properties* (V1), (V1b), *and* (V2) *and $p_0, p_1, \dots$ satisfy conditions* (W1), (W1b), *and* (W2), *then the family of operators $C_R^{(n)} = p_n(C_R)$, $R > 0$, $n = 0, 1, \dots$, has properties* (P1), (P1b), *and* (P2).

Proof. Properties (P1) and (P2) can be established in just the same way as in the proof of Proposition 2.4 in [5]. It must be noted that one cannot immediately refer to this proposition in [5] because conditions (V1a) and (W1a) used there are distinct from conditions (V1b) and (W1b) in the present paper. However, in the verification of properties (P1) and (P2) one uses only conditions (V1), (V2) and (W1), (W2).

We now verify property (P1b). We use the same notation as in the statements of properties (V1), (V1b), (V2) and (W1), (W1b), (W2). Without loss of generality one can assume that the constant R_0 in conditions (V1) and (V1b) is the same, the constant ε in conditions (W1) and (W1b) is the same, and that $R_0^{3/2} \leqslant \varepsilon < 1$ and $\max(I_1 R_0^{1/2}, I_5 R_0^{3/2}, I_0 I_5 R_0^2) < 1$. Moreover, by property (P1) one can assume that for $0 < R \leqslant R_0$ one has the estimate $\|p_n(C_R)\| \leqslant I_7 \exp(-I_8 n R^2)$ for each integer $n \geqslant 0$, where I_7 and I_8 are positive constants.

Let $R \in (0, R_0]$. Since the operator C_R is self-adjoint and $\|C_R\| < 1$, the spectrum of this operator lies in the interior of the interval $[-1, 1]$. It follows by the spectral theorem that the Hilbert space $L_2(S^1)$ can be decomposed into a direct sum of subspaces H_R^+ and H_R^- invariant with respect to the operator C_R, and the spectra of the restrictions of the operator C_R to these subspaces lie in the intervals $[1-R^{3/2},1]$ and $[-1,1-R^{3/2}]$, respectively. Moreover, the spaces H_R^+ and H_R^- are invariant with respect to each bounded operator commuting with C_R. In particular, these spaces are invariant with respect to multiplication by the function g. The function h_R can be represented in the form $h_R^+ + h_R^-$, where $h_R^+ \in H_R^+$ and $h_R^- \in H_R^-$. Moreover, $C_R h_R^+ - (1-gR^2)h_R^+ \in H_R^+$ and $C_R h_R^- - (1-gR^2)h_R^- \in H_R^-$, which yields

$$\|C_R h_R^+ - (1-gR^2)h_R^+\| \leqslant \|C_R h_R - (1-gR^2)h_R\| \leqslant I_0 R^3.$$

Further, $(C_R h_R^+, h_R^+) \leqslant (h_R^+, h_R^+)$ and $(C_R h_R^-, h_R^-) \leqslant (1-R^{3/2})(h_R^-, h_R^-)$, therefore

$$(C_R h_R, h_R) = (C_R h_R^+, h_R^+) + (C_R h_R^-, h_R^-) \leqslant (h_R, h_R) - R^{3/2}(h_R^-, h_R^-).$$

At the same time $|(C_R h_R, h_R) - ((1-gR^2)h_R, h_R)| \leqslant I_0 R^3 \|h_R\|$, and therefore

$$(C_R h_R, h_R) \geqslant (1 - I_0 R^2)(h_R, h_R) - I_0 R^3 \|h_R\|.$$

Consequently,

$$\|h_R^-\|^2 \leqslant I_0 R^{1/2} \|h_R\|^2 + I_0 R^{3/2} \|h_R\|.$$

Since $\|h_R\| \leqslant \|1\| + \|h_R - 1\| \leqslant (2\pi)^{1/2} + I_0 R_0$, we arrive at an estimate of the form $\|h_R^-\| \leqslant I_9 R^{1/4}$, where I_9 is a positive constant that can be expressed in terms of I_0 and R_0. Finally, $\|h_R^+ - 1\| \leqslant \|h_R - 1\| + \|h_R^-\| \leqslant (I_0 R_0^{3/4} + I_9)R^{1/4}$. Obviously, there exists $R_1 \in (0, R_0]$ such that $\|h_R^+ - 1\| < \|1\|$ for $0 < R \leqslant R_1$. Moreover, $h_R^+ \neq 0$. In particular, the space H_R^+ is non-trivial.

Let $R \in (0, R_1]$. We denote by C_R^+ the restriction of the operator C_R to the subspace H_R^+. We set $D_R = 1 - I_5(1 - C_R^+)$. Since $\|C_R\| \leqslant 1 - I_1 R^2$, it follows that the spectrum σ_R^+ of C_R^+ lies in the interval $[1-R^{3/2}, 1-I_1R^2]$ (where $1-R^{3/2} < 1-I_1R^2$ because $I_1 R_0^{1/2} < 1$). Then the spectrum of the operator D_R lies in the interval $[1 - I_5 R^{3/2}, 1 - I_1 I_5 R^2] \subset [0,1]$ and, in particular, $\|D_R\| \leqslant 1 - I_1 I_5 R^2$. We note that

$$\|D_R h_R^+ - (1 - I_5 g R^2) h_R^+\| = I_5 \|C_R^+ h_R^+ - (1 - gR^2) h_R^+\| \leqslant I_0 I_5 R^3.$$

Since the operator of multiplication by the function g commutes with C_R and leaves invariant the space H_R^+, it commutes with the operator D_R. Hence

$$\|D_R^m h_R^+ - (1 - I_5 g R^2) D_R^{m-1} h_R^+\| \leqslant I_0 I_5 R^3 \|D_R^{m-1}\|$$

for each positive integer m, and $\|D_R^{m-1}\| = \|D_R\|^{m-1}$ because the operator D_R is self-adjoint. By the inequality $I_0 I_5 R_0^2 < 1$ the function $1 - I_5 g R^2$ is positive. In particular,

$$\sup |1 - I_5 g R^2| = 1 - I_5 R^2 \inf g \leqslant 1 - I_0^{-1} I_5 R^2.$$

Finally, for each positive integer n we obtain

$$\begin{aligned}\|D_R^n h_R^+ - (1 - I_5 g R^2)^n h_R^+\| &\leqslant I_0 I_5 R^3 \sum_{j=1}^{n} \|D_R\|^{j-1} \sup|1 - I_5 g R^2|^{n-j} \\ &\leqslant I_0 I_5 R^3 \sum_{j=0}^{n-1} (1 - I_1 I_5 R^2)^j \leqslant I_0 I_1^{-1} R.\end{aligned}$$

Further, the operator $p_n(C_R^+) - I_4 D_R^n$ is self-adjoint on the subspace H_R^+, and its spectrum is the image of the spectrum σ_R^+ under the action of the polynomial $p_n(z) - I_4(1 - I_5(1 - z))^n$. Since $\sigma_R^+ \subset [1 - R^{3/2}, 1] \subset [1 - \varepsilon, 1]$, it follows from condition (W1b) that the spectrum of $p_n(C_R^+) - I_4 D_R^n$ lies in the interval $[-I_6(R^{3/2} + nR^3), I_6(R^{3/2} + nR^3)]$. Moreover, $\|p_n(C_R^+) - I_4 D_R^n\| \leqslant I_6(R^{3/2} + nR^3)$, therefore

$$\|p_n(C_R^+) h_R^+ - I_4(1 - I_5 g R^2)^n h_R^+\| \leqslant I_0 I_1^{-1} I_4 R + I_6(R^{3/2} + nR^3)\|h_R^+\|,$$

and $\|h_R^+\| \leqslant \|h_R\| \leqslant (2\pi)^{1/2} + I_0 R_0$. Thus,

$$\|p_n(C_R^+) h_R^+ - I_4(1 - I_5 g R^2)^n h_R^+\| \leqslant I_{10}(R + nR^3),$$

where I_{10} is a positive constant independent of R and n. Hence

$$\|p_n(C_R)1 - I_4(1 - I_5 g R^2)^n\| \leqslant I_{10}(R + nR^3) + (I_4 + I_7)\|h_R^+ - 1\| \leqslant I_{11}(R^{1/4} + nR^3),$$

where $I_{11} = \max(I_{10}, I_{10} R_0^{3/4} + (I_4 + I_7)(I_0 R_0^{3/4} + I_9))$. Moreover,

$$\begin{aligned}\|p_n(C_R)1 - I_4(1 - I_5 g R^2)^n\| &\leqslant (2\pi)^{1/2}(\|p_n(C_R)\| + I_4 \sup|1 - I_5 g R^2|^n) \\ &\leqslant I_{12} \exp(-I_{13} n R^2),\end{aligned}$$

where $I_{12} = (2\pi)^{1/2}(I_4 + I_7)$ and $I_{13} = \min(I_8, I_0^{-1} I_5)$.

For each positive integer n we now define a quantity $r_n > 0$ by the equality $r_n = \exp(-n r_n^2)$. Obviously, $r_n \to 0$ as $n \to \infty$. If $r_n < R_1$, then, as follows from the above estimates,

$$\begin{aligned}n \int_0^{R_1} &R\|p_n(C_R)1 - I_4(1 - I_5 g R^2)^n\|\, dR \\ &\leqslant n \int_0^{r_n} I_{11} R(R^{1/4} + nR^3)\, dR + n \int_{r_n}^{R_1} I_{12} R \exp(-I_{13} n R^2)\, dR \\ &\leqslant \frac{4}{9} I_{11} n r_n^{9/4} + \frac{1}{5} I_{11} n^2 r_n^5 + \frac{1}{2} I_{12} I_{13}^{-1} \exp(-I_{13} n r_n^2).\end{aligned}$$

Since $n r_n^2 = -\log r_n$, we conclude that $n r_n^2 \to \infty$, $n r_n^{9/4} \to 0$, and $n^2 r_n^5 \to 0$ as $n \to \infty$. Thus, the sequence of functions

$$n \int_0^{R_1} R(C_R^{(n)} 1 - I_4(1 - I_5 g R^2)^n)\, dR, \qquad n = 1, 2, \ldots,$$

approaches zero in the space $L_2(S^1)$. At the same time the sequence of functions

$$n\int_0^{R_1} R(1-I_5gR^2)^n\,dR = \frac{n}{2(n+1)I_5g}(1-(1-I_5gR_1^2)^{n+1}), \qquad n=1,2,\dots,$$

approaches $(2I_5g)^{-1}$ uniformly as $n\to\infty$. This means that

$$n\int_0^{R_1} R(C_R^{(n)}1,1)\,dR \to (I_4(2I_5g)^{-1},1) \quad \text{as } n\to\infty,$$

where

$$(I_4(2I_5g)^{-1},1) = I_4\int_{S^1}(2I_5g(t))^{-1}\,dt \geqslant \pi I_0^{-1}I_4I_5^{-1} > 0.$$

The property (P1b) is thus established.

Proposition 2.5. *Let* $\{C_R\}_{R>0}$ *be a family of contracting operators in* $L_2(S^1)$ *continuously dependent on the parameter* R. *If this family has properties* (V1′), (V1b), *and* (V2′), *then the family of operators* $C_R^{(n)} = C_R^n$, $R>0$, $n=0,1,\dots,$ *has properties* (P1), (P1b), *and* (P2).

Proof. Since the operator C_R is contracting, $\|C_R\|\leqslant 1$, for each positive integer m and each integer $n\geqslant 0$ we have $\|C_R^n\|\leqslant \|C_R^m\|^{\lfloor n/m\rfloor}$, where $\lfloor n/m\rfloor$ is the integer part of the ratio n/m. By property (V1′) one can choose m such that $\|C_R^m\|\leqslant 1-I_1R^2$ for $0<R\leqslant R_0$, where I_1 and R_0 are positive constants. Reducing the constant I_1 if necessary we can assume that $1-I_1R_0^2\geqslant 1/2$. Then for $0<R\leqslant R_0$ we have the estimate

$$\|C_R^n\|\leqslant (1-I_1R^2)^{\lfloor n/m\rfloor}\leqslant 2(1-I_1R^2)^{n/m}\leqslant 2(1-m^{-1}I_1R^2)^n.$$

This proves property (P1).

By property (V2′), for all R_1 and R_2, $0<R_1<R_2$, there exists a positive integer M such that $\|C_R^M\|<1$ for $R_1\leqslant R\leqslant R_2$. The function $R\mapsto\|C_R^M\|$ is continuous because the operator C_R depends continuously on R, and therefore there exists $\rho\in(0,1)$ such that $\|C_R^M\|\leqslant\rho$ for $R_1\leqslant R\leqslant R_2$. Then for each integer $n\geqslant 0$ we have

$$\|C_R^n\|\leqslant\|C_R^M\|^{\lfloor n/M\rfloor}\leqslant\rho^{n/M-1}=\rho^{-1}(\rho^{1/M})^n$$

for $R_1\leqslant R\leqslant R_2$. This proves property (P2).

We shall now use property (V1b). Let I_0 and R_0 be constants, let $R\in(0,R_0]$, and let g and h_R be functions mentioned in the definition of this property. By property (P1), which has already been established for small values of R, we have the bound $\|C_R^n\|\leqslant I_2(1-I_3R^2)^n$, where I_2 and I_3 are positive constants. Reducing the constant R_0 if necessary we can assume that this bound holds for $0<R\leqslant R_0$ and that $R_0^2\max(I_0,I_3)<1$. Let $R\in(0,R_0]$. Then $0<1-I_3R^2\leqslant\exp(-I_3R^2)$. Moreover, $0<1-gR^2\leqslant 1-I_0^{-1}R^2\leqslant\exp(-I_0^{-1}R^2)$ on the entire circle. Since the operator C_R commutes with the operator of multiplication by the function g, it follows that $\|C_R^m h_R - (1-gR^2)C_R^{m-1}h_R\| \leqslant I_0R^3\|C_R^{m-1}\|$ for each positive integer m. Hence for each positive integer n we have

$$\|C_R^n h_R - (1-gR^2)^n h_R\| \leqslant I_0R^3\sum_{j=1}^{n}\|C_R^{j-1}\| \leqslant I_0I_2R^3\sum_{j=0}^{n-1}(1-I_3R^2)^j \leqslant I_0I_2I_3^{-1}R.$$

This means that $\|C_R^n 1 - (1-gR^2)^n\| \leqslant I_4R$, where $I_4 = 2I_0 + I_0I_2I_3^{-1}$. Moreover, we have

$$\|C_R^n 1 - (1-gR^2)^n\| \leqslant (2\pi)^{1/2}\big(\|C_R^n\| + \sup|(1-gR^2)^n|\big) \leqslant I_5\exp(-I_6nR^2),$$

where $I_5 = (2\pi)^{1/2}(I_2+1)$ and $I_6 = \min(I_3, I_0^{-1})$. We now define $r_n > 0$ by the equality $I_4r_n = I_5\exp(-I_6nr_n^2)$. Obviously, $r_n \to 0$ as $n \to \infty$. If $r_n < R_0$, then, as follows from the above estimates,

$$\begin{aligned} n\left\|\int_0^{R_0} R(C_R^n 1 - (1-gR^2)^n)\,dR\right\| &\leqslant n\int_0^{R_0} R\|C_R^n 1 - (1-gR^2)^n\|\,dR \\ &\leqslant n\int_0^{r_n} I_4R^2\,dR + n\int_{r_n}^{R_0} I_5R\exp(-I_6nR^2)\,dR \\ &\leqslant \frac{1}{3}I_4nr_n^3 + \frac{1}{2}I_5I_6^{-1}\exp(-I_6nr_n^2). \end{aligned}$$

Since $nr_n^2 = -I_6^{-1}\log(I_4I_5^{-1}r_n)$, it follows that $nr_n^2 \to \infty$ and $nr_n^3 \to 0$ as $n \to \infty$. Thus, the sequence of functions

$$n\int_0^{R_0} R(C_R^n 1 - (1-gR^2)^n)\,dR, \qquad n = 1, 2, \dots,$$

approaches zero in the space $L_2(S^1)$. At the same time, the sequence of functions

$$n\int_0^{R_0} R(1-gR^2)^n\,dR = \frac{n}{2(n+1)g}(1-(1-gR_0^2)^{n+1}), \qquad n = 1, 2, \dots,$$

uniformly converges to the function $(2g)^{-1}$ as $n \to \infty$. Hence

$$n\int_0^{R_0} R(C_R^n 1, 1)\,dR \to ((2g)^{-1}, 1) \quad \text{as } n \to \infty,$$

where $((2g)^{-1}, 1) = \displaystyle\int_{S^1}(2g(t))^{-1}\,dt \geqslant \pi I_0^{-1} > 0$. Property (P1b) is now established.

§ 3. Actions of groups and semigroups

We require three lemmas proved in [5].

Lemma 3.1 [5]. *A su exists no non-trivial lin of H.*

Lemma 3.2 [5]. *If a generated by the vecto*

Lemma 3.3 [5]. *Let 𝒢 common fixed point an is a strip, a square, or*

Let us add two fur group $\mathcal{G}$.

Lemma 3.4. *Let $\mathcal{H}$ b least one rotation thro through angles incomm*

Proof. One must show also belongs to the se order. This holds for a the lemma, for all rota rotation B through so $m > 1$. For an arbitrar the plane through the is of order m. Hence symmetry, then A^2 is contains the isometry

Lemma 3.5. *Let $\mathcal{H}$*

If the semigroup $\mathcal{H}$ co $A_1, \dots, A_k$ satisfy one

Proof. Let P_0 be the We denote by T_v the assume first that each the sum of the vectors condition (i) in Theor Hence the isometries element of $\mathcal{H}$ is invert

We consider now t isometries not belongi $1 \leqslant k_1 \leqslant k$. Since the are axial or translatio be an axial symmetry For $1 \leqslant j \leqslant k$ we de

Lemma 3.1 [5]. *A subgroup H of the group $\mathbb{R}^2$ is dense in $\mathbb{R}^2$ if and only if there exists no non-trivial linear functional on $\mathbb{R}^2$ taking an integer value on each element of H.*

Lemma 3.2 [5]. *If an angle φ is not a multiple of $\pi/2$ or $\pi/3$, then the group generated by the vectors $v_n = (\cos n\varphi, \sin n\varphi)$, $n = 0, 1, 2, \dots,$ is dense in $\mathbb{R}^2$.*

Lemma 3.3 [5]. *Let $\mathcal{H}$ be a subgroup of the group $\mathcal{G}$. If the isometries in $\mathcal{H}$ have no common fixed point and do not preserve a common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle, then the group $\mathcal{H} \cap \mathcal{G}_0$ is dense in $\mathcal{G}_0$.*

Let us add two further lemmas concerning properties of subsemigroups of the group $\mathcal{G}$.

Lemma 3.4. *Let $\mathcal{H}$ be a semigroup of isometries of the plane P. If $\mathcal{H}$ contains at least one rotation through an angle distinct from zero, but does not contain rotations through angles incommensurable with π, then the semigroup $\mathcal{H}$ is a group.*

Proof. One must show that for each isometry $A \in \mathcal{H}$ the inverse transformation A^{-1} also belongs to the semigroup $\mathcal{H}$. This is obvious if the isometry A has a finite order. This holds for all axial symmetries and, as follows from the assumptions of the lemma, for all rotations belonging to $\mathcal{H}$. Further, by assumption, $\mathcal{H}$ contains a rotation B through some non-zero angle. This rotation has some (finite) order m, $m > 1$. For an arbitrary parallel translation $A \in \mathcal{H}$ the isometry AB is a rotation of the plane through the same angle as B (but about another point); in particular, AB is of order m. Hence $A^{-1} = B(AB)^{m-1} \in \mathcal{H}$. Finally, if $A \in \mathcal{H}$ is a translational symmetry, then A^2 is a parallel translation. As already proved, the semigroup $\mathcal{H}$ contains the isometry $(A^2)^{-1}$ and therefore also the isometry $(A^2)^{-1}A = A^{-1}$.

Lemma 3.5. *Let $\mathcal{H}$ be a semigroup generated by isometries*

$$A_1, \dots, A_k \in \mathcal{G}.$$

If the semigroup $\mathcal{H}$ contains no rotations and is not a group, then the isometries $A_1, \dots, A_k$ satisfy one of conditions (i) *and* (ii) *in Theorem* 1.4.

Proof. Let P_0 be the Euclidean vector space associated with the affine plane P. We denote by T_v the parallel translation of the plane P by the vector $v \in P_0$. We assume first that each isometry A_j, $1 \leqslant j \leqslant k$, has the form T_{v_j}, where $v_j \in P_0$. If the sum of the vectors $v_1, \dots, v_k$ is non-zero, then the isometries $A_1, \dots, A_k$ satisfy condition (i) in Theorem 1.4. Otherwise $A_1 A_2 \dots A_k$ is the identity transformation. Hence the isometries $A_1, \dots, A_k$ are invertible in the semigroup $\mathcal{H}$. Then each element of $\mathcal{H}$ is invertible in $\mathcal{H}$, that is, $\mathcal{H}$ is a group.

We consider now the case when among the isometries $A_1, \dots, A_k$ there exist isometries not belonging to $\mathcal{G}_0$. For definiteness let these isometries be $A_1, \dots, A_{k_1}$, $1 \leqslant k_1 \leqslant k$. Since the semigroup $\mathcal{H}$ does not contain rotations, all these isometries are axial or translational symmetries with axes parallel to some fixed line l. Let S be an axial symmetry with respect to l. We consider an arbitrary point $x \in l$. For $1 \leqslant j \leqslant k$ we denote by v_j the vector $A_j x - x \in P_0$. Then $A_j = T_{v_j} S$ for

$1 \leqslant j \leqslant k_1$ and $A_j = T_{v_j}$ for $k_1 < j \leqslant k$. The transformation $A_1^2 A_2^2 \dots A_k^2$ is the parallel translation by the vector

$$w = \sum_{1 \leqslant j \leqslant k_1} (v_j + Sv_j) + 2 \sum_{k_1 < j \leqslant k} v_j,$$

where Sv_j is the vector symmetric to v_j with respect to the line l. The orthogonal projection of the vector $w/2$ onto the direction of the line l is equal to the projection of the sum of the vectors $v_1, \dots, v_k$. Thus, if the vector w is not orthogonal to l, then the isometries $A_1, \dots, A_k$ satisfy condition (ii) in Theorem 1.4. If the vector w is orthogonal to l, then $ST_w = T_{-w}S$, and therefore $T_w A_1 T_w = A_1$. As a consequence, $A_1 A_2^2 \dots A_k^2 A_1^3 A_2^2 \dots A_k^2$ is the identity transformation. Thus, the semigroup $\mathcal{H}$ contains the isometry

$$A_k^{-2} \dots A_2^{-2} A_1^{-3} A_k^{-2} \dots A_2^{-2} A_1^{-1}.$$

Hence $\mathcal{H}$ contains the isometries $A_1^{-1}, \dots, A_k^{-1}$, which means that $\mathcal{H}$ is a group.

We now discuss the degenerate cases (i) and (ii) in Theorem 1.4.

Lemma 3.6. *If isometries $A_1, \dots, A_k \in \mathcal{G}$ satisfy one of conditions* (i) *and* (ii) *in Theorem* 1.4, *then the action $d = \mathrm{FSG}_k[A_1, \dots, A_k]$ satisfies condition* (2) *in the same theorem.*

Proof. We prove an auxiliary combinatorial result first. Let $a_1, \dots, a_k$ be the generators of the semigroup FSG_k. An arbitrary element $g \in \mathrm{FSG}_k$ of length $n > 0$ has a unique representation of the form $a_{j_1} a_{j_2} \cdots a_{j_n}$, where $1 \leqslant j_s \leqslant k$, $s = 1, \dots, n$. For each $j \in \{1, \dots, k\}$ we denote by $m_j(g)$ the number of elements of the sequence $j_1, \dots, j_n$ equal to j. Let $\varepsilon \in (0, k^{-1})$. We denote by $\widetilde{N}_{\varepsilon,j}^{(n)}$, $1 \leqslant j \leqslant k$, the ratio of the number of elements g of length n for which $m_j(g) \leqslant (k^{-1} - \varepsilon)n$ to the number k^n of all elements of length n in FSG_k and we denote by $\widetilde{N}_\varepsilon^{(n)}$ the ratio of the number of elements $g \in \mathrm{FSG}_k$ of length n for which at least one of the numbers $m_1(g), \dots, m_k(g)$ does not exceed $(k^{-1} - \varepsilon)n$ to k^n. Obviously, the quantities $\widetilde{N}_{\varepsilon,1}^{(n)}, \dots, \widetilde{N}_{\varepsilon,k}^{(n)}$ are equal, and $\widetilde{N}_\varepsilon^{(n)} \leqslant k \widetilde{N}_{\varepsilon,1}^{(n)}$. One can readily see that

$$\widetilde{N}_{\varepsilon,1}^{(n)} = k^{-n} \sum_{0 \leqslant j \leqslant (k^{-1} - \varepsilon)n} \binom{n}{j} (k-1)^{n-j},$$

where $\binom{n}{j} = \dfrac{n!}{j!\,(n-j)!}$ is the binomial coefficient. We note that the inequality $j \leqslant (k^{-1} - \varepsilon)n$ leads to the inequality $(k - 1 + k\varepsilon)j \leqslant (1 - k\varepsilon)(n - j)$. This yields the following inequality for an arbitrary $\delta > 0$:

$$\begin{aligned} \widetilde{N}_{\varepsilon,1}^{(n)} &\leqslant k^{-n} \sum_{0 \leqslant j \leqslant (k^{-1} - \varepsilon)n} \binom{n}{j} e^{-\delta(k-1+k\varepsilon)j} e^{\delta(1-k\varepsilon)(n-j)} (k-1)^{n-j} \\ &\leqslant k^{-n} \big(e^{-\delta(k-1+k\varepsilon)} + (k-1)e^{\delta(1-k\varepsilon)}\big)^n. \end{aligned}$$

If δ is sufficiently small, then $e^{-\delta(k-1+k\varepsilon)}+(k-1)e^{\delta(1-k\varepsilon)}$ is less than k. Thus, there exists $I_\varepsilon>0$ such that the sequence $\{\exp(I_\varepsilon n)\widetilde{N}_{\varepsilon,1}^{(n)}\}$ is bounded. Moreover, the sequence $\{\exp(I_\varepsilon n)\widetilde{N}_{\varepsilon}^{(n)}\}$ is also bounded.

Let $A_1,\dots,A_k$ be isometries satisfying condition (i). All of them are parallel translations by some vectors $v_1,\dots,v_k$, and the sum $v=v_1+\dots+v_k$ is distinct from zero. We now choose $\varepsilon\in(0,k^{-1})$ such that $2k^2\varepsilon(|v_1|+\dots+|v_k|)\leqslant|v|$. Let $x\in P$ and let E be a bounded subset of P. We consider a positive integer n_0 such that the distance from the point x to any point in the set E is not greater than $n_0|v|/(2k)$. Let g be an arbitrary element of FSG_k of length $n\geqslant n_0$. The isometry $d(g)$ is a parallel translation by the vector $w_g=m_1(g)v_1+\dots+m_k(g)v_k$. Assume that $m_j(g)>(k^{-1}-\varepsilon)n$ for $1\leqslant j\leqslant k$. Since $m_1(g)+\dots+m_k(g)=n$, it follows that $|m_j(g)-n/k|<k\varepsilon n$, $j=1,\dots,k$. Hence $|w_g-(n/k)v|<k\varepsilon n(|v_1|+\dots+|v_k|)\leqslant n|v|/(2k)$. Moreover, $|w_g|>n|v|/(2k)$. In particular, $d(g)x\notin E$. Thus, $d(g)x\in E$ only if at least one of the quantities $m_1(g),\dots,m_k(g)$ does not exceed $(k^{-1}-\varepsilon)n$. Consequently, $N_{d,x}^{(n)}(E)\leqslant\widetilde{N}_{\varepsilon}^{(n)}$ for $n\geqslant n_0$ and the sequence $\{\exp(I_\varepsilon n)N_{d,x}^{(n)}(E)\}$ is bounded.

We assume now that the isometries $A_1,\dots,A_k$ satisfy condition (ii). In this case each of them is either a parallel translation or an axial or a translational symmetry with axis parallel to some fixed line l. We consider a point $x\in P$. For each j, $1\leqslant j\leqslant k$, let w_j be the orthogonal projection of the vector A_jx-x to the direction of the line l. We note that the vectors $w_1,\dots,w_k$ do not depend on our choice of the point x and their sum is distinct from zero by condition (ii). We set $d_1=\mathrm{FSG}_k[B_1,\dots,B_k]$, where $B_1,\dots,B_k$ are the parallel translations by the vectors $w_1,\dots,w_k$, respectively. The isometries $B_1,\dots,B_k$ satisfy condition (i), therefore, as already proved, there exists $I_0>0$ such that the sequence $\{\exp(I_0n)N_{d_1,x_0}^{(n)}(E_0)\}$ is bounded for each bounded subset E_0 of P and each point $x_0\in P$. Let $x\in P$ and let E be a bounded subset of P. We denote by E_1 the orthogonal projection of the set E onto the line l and by x_1 the orthogonal projection of the point x onto l. One can readily see that for each $g\in\mathrm{FSG}_k$ the point $d_1(g)x_1$ is the orthogonal projection of the point $d(g)x$ onto l. Thus, $d_1(g)x_1\in E_1$ once $d(g)x\in E$. Hence $N_{d,x}^{(n)}(E)\leqslant N_{d_1,x_1}^{(n)}(E_1)$ for $n=0,1,\dots$. The set E_1 is bounded because so is E. Consequently, the sequence $\{\exp(I_0n)N_{d_1,x_1}^{(n)}(E_1)\}$ is bounded, and therefore so is also the sequence $\{\exp(I_0n)N_{d,x}^{(n)}(E)\}$.

Let $A_1,\dots,A_k\in\mathcal{G}$. We denote by $\mathcal{H}$ the semigroup generated by the isometries $A_1,\dots,A_k$ and set $C=k^{-1}(u[A_1]+\dots+u[A_k])$. Let $\{C_R\}_{R>0}$ be the family of radial operators corresponding to the operator C in some Cartesian coordinate system. The following three lemmas define the conditions under which this family has properties (V1′), (V1b), and (V2′) formulated in § 2.

Lemma 3.7. *If the isometries $A_1,\dots,A_k$ have no common fixed point and the semigroup $\mathcal{H}$ contains a rotation through an angle that is not a multiple of $\pi/2$ or $\pi/3$, then the family of operators C_R, $R>0$, has properties* (V1′) *and* (V2′).

Proof. Let $A\in\mathcal{H}$ be a rotation through an angle that is not a multiple of $\pi/2$ or $\pi/3$. Let x be the centre of the rotation A. We claim that the semigroup $\mathcal{H}$ contains a rotation B not commuting with A, that is, having another centre. If $\mathcal{H}$ contains a non-trivial parallel translation T, then one can set $B = TA$. If $\mathcal{H}$ contains an axial symmetry with axis not passing through the point x, then one can set $B = SAS$. Hence if the semigroup $\mathcal{H}$ does not contain the required rotation, then it consists of rotations and axial symmetries preserving the point x. However, this contradicts the assumptions of the lemma.

Let $d = \mathrm{FSG}_k[A_1, \dots, A_k]$. Since the semigroup $\mathcal{H}$ contains rotations A and B, there exist elements $g_1, g_2 \in \mathrm{FSG}_k$ such that $d(g_1) = A$ and $d(g_2) = B$. We set $m_1 = |g_1|$ and $m_2 = |g_2|$. Let n be a positive integer not exceeding the order of the rotation A. The isometries $A, A^2, \dots, A^{n-1}$ are non-trivial rotations about the point x, and therefore none of them commutes with B. This means that the isometries $A^j B A^{n-1-j}$, $j = 0, 1, \dots, n-1$, are pairwise distinct. The elements $g_1^j g_2 g_1^{n-1-j}$, $j = 0, 1, \dots, n-1$, of the semigroup FSG_k have the same length $m = (n-1)m_1 + m_2$ and are distinct because $d(g_1^j g_2 g_1^{n-1-j}) = A^j B A^{n-1-j}$. Thus, the operator $(kC)^m = \sum_{|g|=m} u[d(g)]$ has the form $nD + Y$, where $D = n^{-1} \sum_{j=0}^{n-1} u[A^j B A^{n-1-j}]$ and Y is the sum of $k^m - n$ operators of the form $u[X]$, $X \in \mathcal{G}$. By Lemma 2.2, $(kC_R)^m = nD_R + Y_R$ for each $R > 0$, where Y_R is the sum of $k^m - n$ unitary operators in $L_2(S^1)$. Hence $1 - \|C_R^m\| \geqslant nk^{-m}(1 - \|D_R\|)$. The rotation B can be represented in the form $T_0 B_0$, where T_0 is a non-trivial parallel translation and B_0 is a rotation about the point x. Moreover, we have $nD = u[A^{n-1}B_0] \sum_{j=0}^{n-1} u[A^j T_0 A^{-j}]$. In the Cartesian coordinate system in which we calculate the radial operators T_0 is a parallel translation by a non-trivial vector $v \in \mathbb{R}^2$. Moreover, the isometry $A^j T_0 A^{-j}$ is a parallel translation by the vector $A_0^j v$, where A_0 is the homogeneous part of the rotation A. Then for each $R > 0$ the radial operator $u[A^{n-1}B_0]_R^{-1} D_R = n^{-1} \sum_{j=0}^{n-1} u[A^j T_0 A^{-j}]_R$ is the operator of multiplication by the function $h_{n,R} = n^{-1} \sum_{j=0}^{n-1} \exp(iR(\Phi, A_0^j v))$. Since the operator $u[A^{n-1}B_0]_R$ is unitary, it follows that $\|D_R\| = \sup |h_{n,R}|$. Finally, $1 - \|C_R^m\| \geqslant nk^{-m}(1 - \sup |h_{n,R}|)$.

We now choose $I_1 > 0$ such that $|e^{iy} - (1 + iy - y^2/2)| \leqslant I_1 |y|^3$ for $|y| \leqslant 1$. Then

$$\sup |h_{n,R} - (1 + iRh_n^{(1)} - R^2 h_n^{(2)})| \leqslant I_1 |v|^3 R^3$$

for $0 < R \leqslant |v|^{-1}$ and each positive integer n, where

$$h_n^{(1)} = \frac{1}{n} \sum_{j=0}^{n-1} (\Phi, A_0^j v), \qquad h_n^{(2)} = \frac{1}{2n} \sum_{j=0}^{n-1} (\Phi, A_0^j v)^2.$$

We set $I_0 = \inf((\Phi, v)^2 + (\Phi, A_0 v)^2)$. The vectors v and $A_0 v$ are non-collinear, and therefore the inner products $(\Phi(t), v)$ and $(\Phi(t), A_0 v)$ cannot vanish simultaneously. Since the map Φ is continuous, it follows that $I_0 > 0$. For each $t \in S^1$ and each integer $j \geqslant 0$ we have the equality $(\Phi(t), A_0^j v) = (\Phi(t - j\varphi), v)$, where φ is the angle of the rotation A. This implies that $\inf((\Phi, A_0^j v)^2 + (\Phi, A_0^{j+1} v)^2) = I_0$ for $j = 0, 1, \dots$. Thus, if $n \geqslant 2$, then

$$\inf \sum_{j=0}^{n-1} (\Phi, A_0^j v)^2 \geqslant \frac{(n-1) I_0}{2} \geqslant \frac{n I_0}{4},$$

that is, $\inf h_n^{(2)} \geqslant I_0/8$. We observe that the ratio of the lengths of the vectors $v_n = \sum_{j=0}^{n-1} A_0^j v$ and v is equal to $|1 + e^{i\varphi} + \cdots + e^{i(n-1)\varphi}| = |1 - e^{in\varphi}|/|1 - e^{i\varphi}|$, so that the sequence $v_1, v_2, \ldots$ contains vectors of an arbitrarily small length. If the rotation A has finite order, then we set n to be equal to this order, and then $v_n = 0$. If the order of A is infinite, then we take some $n \geqslant 2$ such that $|v_n| \leqslant I_0^{1/2}/4$. In any case $\sup |h_n^{(1)}| = n^{-1}|v_n| \leqslant I_0^{1/2}/4$. Further, $\inf h_n^{(2)} \geqslant I_0/8$ because $n \geqslant 2$. Finally, n does not exceed the order of A, therefore one can find a positive integer m and a constant $I_2 > 0$ such that $1 - \|C_R^m\| \geqslant I_2(1 - \sup |h_{n,R}|)$ for each $R > 0$. Consider arbitrary $R \in (0, |v|^{-1})$. Since $\sup h_n^{(2)} \leqslant |v|^2/2$, the function $1 - R^2 h_n^{(2)}$ is positive. Moreover,

$$\sup |1 - R^2 h_n^{(2)}|^2 \leqslant \sup |1 - R^2 h_n^{(2)}| = 1 - R^2 \inf h_n^{(2)} \leqslant 1 - \frac{I_0 R^2}{8}\,.$$

Since the functions $h_n^{(1)}$ and $h_n^{(2)}$ are real-valued, it follows that

$$\sup |1 + iRh_n^{(1)} - R^2 h_n^{(2)}|^2 = \sup |1 - R^2 h_n^{(2)}|^2 + R^2 \sup |h_n^{(1)}|^2 \leqslant 1 - \frac{I_0 R^2}{16}\,,$$

and therefore $\sup |1 + iRh_n^{(1)} - R^2 h_n^{(2)}| \leqslant 1 - I_0 R^2/32$. Finally,

$$\sup |h_{n,R}| \leqslant \sup |1 + iRh_n^{(1)} - R^2 h_n^{(2)}| + I_1 |v|^3 R^3 \leqslant 1 - \frac{I_0 R^2}{32} + I_1 |v|^3 R^3.$$

Hence the estimate $\sup |h_{n,R}| \leqslant 1 - I_0 R^2/64$ holds for sufficiently small values of R; moreover, $\|C_R^m\| \leqslant 1 - I_0 I_2 R^2/64$. This proves property (V1′).

We now consider arbitrary $R_1 > 0$. The vectors $A_0 v - v$ and $A_0(A_0 v - v)$ are non-collinear, therefore each linear functional on $\mathbb{R}^2$ taking integer values at these vectors is an integer linear combination of the functionals l_0 and l_1 defined by the relations $l_0(A_0 v - v) = l_1(A_0(A_0 v - v)) = 1$ and $l_0(A_0(A_0 v - v)) = l_1(A_0 v - v) = 0$. Only finitely many functionals of this kind have norms not exceeding $(2\pi)^{-1} R_1$. Since A is a rotation through an angle that is not a multiple of $\pi/2$ or $\pi/3$, it follows from Lemma 3.2 that the vectors $A_0^{j+1} v - A_0^j v = A_0^j(A_0 v - v)$, $j = 0, 1, \ldots$, generate a subgroup of the group $\mathbb{R}^2$ that is dense in $\mathbb{R}^2$. It now follows from Lemma 3.1 that an arbitrary non-trivial linear functional on the space $\mathbb{R}^2$ takes a non-integer value at some vector $A_0^{j+1} v - A_0^j v$, $j \geqslant 0$. Thus, there exists a positive integer n such that each non-trivial linear functional on $\mathbb{R}^2$ of norm not exceeding $(2\pi)^{-1} R_1$ takes a non-integer value at least at one of the $n - 1$ vectors $A_0^{j+1} v - A_0^j v$, $j = 0, 1, \ldots, n-2$. If A has finite order n_0, then one can choose n to be not greater than $n_0 + 1$ because in this case $A_0^{n_0+j+1} v - A_0^{n_0+j} v = A_0^{j+1} v - A_0^j v$ for each integer j, $j \geqslant 0$. Moreover, one can choose n to be not greater than n_0 because the sum of the vectors $A_0^{j+1} v - A_0^j v$, $j = 0, 1, \ldots, n_0 - 1$, vanishes. Furthermore, as shown above, one can find a positive integer M and a constant $I_3 > 0$ such that the inequality $1 - \|C_R^M\| \geqslant I_3(1 - \sup |h_{n,R}|)$ holds for each $R > 0$. If $|h_{n,R}(t)| = 1$ for some $t \in S^1$, then the quantities $\exp(iR(\Phi(t), A_0^j v))$, $j = 0, 1, \ldots, n-1$, are all equal and the numbers $(2\pi)^{-1} R(\Phi(t), A_0^{j+1} v - A_0^j v)$, $j = 0, 1, \ldots, n-2$, are integers. The map $w \mapsto (2\pi)^{-1}R(\Phi(t), w)$ is a non-trivial linear functional on $\mathbb{R}^2$ of norm $(2\pi)^{-1}R$. Thus, for $0 < R \leqslant R_1$ we have $|h_{n,R}| < 1$ on the entire circle. Since the function $h_{n,R}$ is continuous, $\sup|h_{n,R}| < 1$ for $0 < R \leqslant R_1$. Hence $\|C_R^M\| < 1$ for $0 < R \leqslant R_1$. Since our choice of R_1 can be arbitrary, the family of operators $\{C_R\}_{R>0}$ has property (V2′).

Lemma 3.8. *If the isometries $A_1, \dots, A_k$ have no common fixed point and do not preserve a common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle, and if the semigroup $\mathcal{H}$ is a group, then the family of operators C_R, $R > 0$, has properties* (V1′) *and* (V2′).

Proof. Let $d = \mathrm{FSG}_k[A_1, \dots, A_k]$. Since the semigroup $\mathcal{H}$ is a group, there exists an element $g_0 \in \mathrm{FSG}_k$ of positive length m_0 such that $d(g_0)$ is the identity transformation.

The Cartesian coordinate system used for the calculation of the radial operators C_R, $R > 0$, enables one to regard an arbitrary element of the group $\mathcal{G}_0$ as a parallel translation by some vector $v \in \mathbb{R}^2$; we denote this element by T_v. We set $H_0 = \{v \in \mathbb{R}^2 \mid T_v \in \mathcal{H}\}$ and denote by H_1 the set of vectors $v \in \mathbb{R}^2$ such that each of the parallel translations T_v and T_{-v} can be represented in the form $d(g)$, where g is an element of the semigroup FSG_k of length $|g|$ divisible by m_0. The set H_0 is a subgroup of $\mathbb{R}^2$ because $\mathcal{H}$ is a group. The set H_1 is also a subgroup of $\mathbb{R}^2$, which immediately follows from its definition. By Lemma 3.3, the group $\mathcal{H} \cap \mathcal{G}_0$ is dense in $\mathcal{G}_0$, therefore the group H_0 is dense in $\mathbb{R}^2$. The group H_1 is also dense in $\mathbb{R}^2$ because it contains the group $m_0H_0 = \{m_0v \mid v \in H_0\}$.

Let $v_1, \dots, v_n \in H_1$ be vectors such that the family $v_1, -v_1, \dots, v_n, -v_n$ contains no equal vectors. According to the definition of the set H_1, one can find elements $g_1^+, g_1^-, \dots, g_n^+, g_n^-$ of the semigroup FSG_k such that $d(g_j^+) = T_{v_j}$, $d(g_j^-) = T_{-v_j}$, $j = 1, \dots, n$, and the lengths of these elements are divisible by m_0. Since $d(g_0^p g) = d(g)$ for each $g \in \mathrm{FSG}_k$ and each positive integer p, where $|g_0^p g| = pm_0 + |g|$, the elements $g_1^+, g_1^-, \dots, g_n^+, g_n^-$ can be chosen having the same length $m > 0$. Moreover, the selected elements are all distinct because so are the parallel translations $T_{v_1}, T_{-v_1}, \dots, T_{v_n}, T_{-v_n}$. Hence the operator

$$(kC)^m = \sum_{|g|=m} u[d(g)]$$

has the form $2nD + Y$, where $D = (2n)^{-1}\big(u[T_{v_1}] + u[T_{-v_1}] + \dots + u[T_{v_n}] + u[T_{-v_n}]\big)$ and Y is the sum of $k^m - 2n$ operators of the form $u[A]$, $A \in \mathcal{G}$. By Lemma 2.2, $(kC_R)^m = 2nD_R + Y_R$ for each $R > 0$, where Y_R is the sum of $k^m - 2n$ unitary operators in $L_2(S^1)$. As a consequence, $1 - \|C_R^m\| \geqslant 2nk^{-m}(1 - \|D_R\|)$. It now follows from Lemma 2.2 that D_R is the operator of multiplication by the function $h_R = n^{-1}\sum_{j=1}^n \cos(R(\Phi, v_j))$. Moreover, $\|D_R\| = \sup|h_R|$.

Since the group H_1 is dense in $\mathbb{R}^2$, it contains a pair of non-collinear vectors v_1 and v_2. Here the vectors $v_1, -v_1, v_2, -v_2$ are pairwise distinct and by the above there exist a positive integer m and $I_1 > 0$ such that $\|C_R^m\| \leqslant 1 - I_1(1 - \sup|h_R|)$ for each $R > 0$, where $h_R = (\cos(R(\Phi, v_1)) + \cos(R(\Phi, v_2)))/2$. We now choose a constant $R_0 > 0$ such that $R_0|v_1| \leqslant \pi/2$ and $R_0|v_2| \leqslant \pi/2$. For $0 < R \leqslant R_0$ we have $0 \leqslant h_R \leqslant 1 - \frac{1}{4}R^2((\Phi, v_1)^2 + (\Phi, v_2)^2)$ on the entire circle S^1.

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Then $1 - \sup|h_R| \geqslant I_0 R^2/4$, where $I_0 = \inf((\Phi, v_1)^2 + (\Phi, v_2)^2)$. Since the vectors v_1 and v_2 are non-collinear, the inner products $(\Phi(t), v_1)$ and $(\Phi(t), v_2)$ cannot simultaneously vanish. As a consequence, the constant I_0 is positive. Moreover, the family $\{C_R\}_{R>0}$ has property (V1′).

We consider now arbitrary $R_1 > 0$. Each linear functional on $\mathbb{R}^2$ taking integer values at the vectors v_1 and v_2 is an integer linear combination of the functionals l_1 and l_2 defined by the relations $l_1(v_1) = l_2(v_2) = 1$ and $l_1(v_2) = l_2(v_1) = 0$. Only finitely many such functionals have norm not exceeding $\pi^{-1}R_1$. It follows from Lemma 3.1 that an arbitrary non-trivial linear functional on the space $\mathbb{R}^2$ takes a non-integer value at some vector $v \in H_1$. Thus, there exist vectors $v_3, \dots, v_n \in H_1$ such that each non-trivial linear functional on $\mathbb{R}^2$ of norm not exceeding $\pi^{-1}R_1$ takes a non-integer value at least at one of the vectors $v_1, \dots, v_n$. These vectors can be chosen so that the family $v_1, -v_1, \dots, v_n, -v_n$ contains no equal vectors. As follows from the above, there exists a positive integer M and a constant $I_2 > 0$ such that $1 - \|C_R^M\| \geqslant I_2(1 - \sup|\widetilde{h}_R|)$ for each $R > 0$, where $\widetilde{h}_R = n^{-1}\sum_{j=1}^{n}\cos(R(\Phi, v_j))$. If $|\widetilde{h}_R(t)| = 1$ for some $t \in S^1$, then the numbers $\pi^{-1}R(\Phi(t), v_1), \dots, \pi^{-1}R(\Phi(t), v_n)$ are integers. The map $v \mapsto \pi^{-1}R(\Phi(t), v)$ is a non-trivial linear functional of norm $\pi^{-1}R$. Thus, for $0 < R \leqslant R_1$ we have $|\widetilde{h}_R| < 1$ on the entire circle. Since the function $\widetilde{h}_R$ is continuous, it follows that $\sup|\widetilde{h}_R| < 1$ for $0 < R \leqslant R_1$. Hence $\|C_R^M\| < 1$ for $0 < R \leqslant R_1$. Since R_1 can be arbitrary, the family of operators $\{C_R\}_{R>0}$ has property (V2′).

Lemma 3.9. *If the isometries $A_1, \dots, A_k$ do not satisfy conditions* (i) *and* (ii) *of Theorem* 1.4 *and the family of operators C_R, $R > 0$, has property* (V1′), *then this family has property* (V1b).

Proof. By property (V1′), there exist positive constants I_1 and R_0 and a positive integer m such that $\|C_R^m\| \leqslant 1 - I_1 R^2$ for $0 < R \leqslant R_0$.

The Cartesian coordinate system in which we calculated the radial operators C_R, $R > 0$, enables one to regard an arbitrary isometry in $\mathcal{G}$ as a transformation of $\mathbb{R}^2$. Each isometry A_j, $1 \leqslant j \leqslant k$, can be represented in the form $A_j = B_j T_j$, where B_j is an isometry preserving the origin and T_j is a parallel translation by some vector $v_j \in \mathbb{R}^2$. The map $b_j = \Phi^{-1}B_j\Phi$ is a well-defined transformation of the circle S^1 of the form $t \mapsto t+\varphi$ or $t \mapsto \varphi - t$, where $\varphi \in S^1$. We denote by U_j the unitary operator in the space $L_2(S^1)$ acting by the rule $h \mapsto h \circ b_j$. By Lemma 2.2, $u[B_j]_R = U_j$ for each $R > 0$, and the radial operator $u[T_j]_R$ is the operator of multiplication by the function $e^{iR(\Phi, v_j)}$. Hence $C_R h = k^{-1}\sum_{j=1}^{k} e^{iR(\Phi, v_j)}U_j h$ for each $R > 0$ and each function $h \in L_2(S^1)$. We now define the operators D_0, D_1, and D_2 on the space $L_2(S^1)$ by the formulae $D_0 h = k^{-1}\sum_{j=1}^{k} U_j h$, $D_1 h = k^{-1}\sum_{j=1}^{k}(\Phi, v_j)U_j h$, and $D_2 h = -(2k)^{-1}\sum_{j=1}^{k}(\Phi, v_j)^2 U_j h$, $h \in L_2(S^1)$. Obviously, there exists a constant $I_2 > 0$ such that $\|C_R - (D_0 + iRD_1 + R^2 D_2)\| \leqslant I_2 R^3$ for $0 < R \leqslant R_0$.

Let H_1, H_2, and H_2^+ be the following real linear subspaces of the space $L_2(S^1)$. The first space is spanned by the functions $t \mapsto \cos t$ and $t \mapsto \sin t$, the second by the functions $t \mapsto \cos 2t$ and $t \mapsto \sin 2t$, and the third by the function 1 and the space H_2. One can readily see that the spaces H_1, H_2, and H_2^+ are invariant with respect to the operator D_0. An arbitrary non-trivial function $f \in H_1$ attains its maximum at a unique point $t_0 \in S^1$. For each integer j, $1 \leqslant j \leqslant k$, the relation

$U_j f = f$ holds if and only if $b_j(t_0) = t_0$. The equality $D_0 f = f$ holds if and only if t_0 is a common fixed point of the maps $b_1, \dots, b_k$. Further, a non-trivial function $f \in H_2$ takes its maximum at two points $t_1, t_2 \in S^1$, and the distance between t_1 and t_2 is π. Moreover, the equality $U_j f = f$, $1 \leqslant j \leqslant k$, holds if and only if the map b_j takes the set $\{t_1, t_2\}$ to itself. The equality $D_0 f = f$ holds if and only if the set $\{t_1, t_2\}$ is invariant with respect to each of the maps $b_1, \dots, b_k$. Thus, if the restriction of the operator $1 - D_0$ to one of the spaces H_1 and H_2 is degenerate, then each of the maps $b_1, \dots, b_k$ either has a fixed point or transposes two points lying at distance π from each other. In either case the maps $b_1, \dots, b_k$ are involutive. Hence the operators $U_1, \dots, U_k$ are self-adjoint, and so is also the operator D_0.

Obviously, $D_1 1 = k^{-1}(\Phi, v_1 + \dots + v_k) \in H_1$. We claim that there exists a function $f_1 \in H_1$ such that $(1 - D_0) f_1 = D_1 1$. This is clear in the case when the restriction of the operator $1 - D_0$ to the space H_1 is non-degenerate. Otherwise the maps $b_1, \dots, b_k$ have a common fixed point $t_0 \in S^1$ and each of the isometries $B_1, \dots, B_k$ is either the identity map or an axial symmetry with axis l passing through the origin and the point $\Phi(t_0)$. We assume first that $B_1, \dots, B_k$ are the identity transformations. If the sum of vectors $v_1, \dots, v_k$ is distinct from zero, then the isometries $A_1, \dots, A_k$ satisfy condition (i) in Theorem 1.4. If this sum is equal to zero, then $D_1 1 = 0$, and one can set $f_1 = 0$. We now proceed to the case when at least one of the isometries $B_1, \dots, B_k$ is an axial symmetry. For an arbitrary point $x \in \mathbb{R}^2$ the orthogonal projection of the vector $A_j x - x$, $1 \leqslant j \leqslant k$, onto the line l is equal to the projection of the vector v_j. Thus, if the sum of the vectors $v_1, \dots, v_k$ is not orthogonal to l, then the isometries $A_1, \dots, A_k$ satisfy condition (ii) in Theorem 1.4. If this sum is orthogonal to l, that is, if $(D_1 1)(t_0) = 0$, then $D_0(D_1 1) = k^{-1}(k - 2k_1) D_1 1$, where k_1 is the number of axial symmetries among the isometries $B_1, \dots, B_k$, and we can set $f_1 = k(2k_1)^{-1} D_1 1$.

Clearly, $D_2 1 \in H_2^+$ and $D_1 H_1 \subset H_2^+$, therefore the function $h_0 = D_2 1 - D_1 f_1$ belongs to the space H_2^+. We denote by $\widetilde{D}$ the restriction of the operator $1 - D_0$ to this space. Let $\widetilde{H}_0$ ($\widetilde{H}_2$) be the kernel (respectively, the range) of the operator $\widetilde{D}$. One can readily see that $1 \in \widetilde{H}_0$ and $\widetilde{D} H_2 = \widetilde{H}_2$. If the restriction of the operator $1 - D_0$ to the space H_2 is non-degenerate, then $\widetilde{H}_2 = H_2$, and the space $\widetilde{H}_0$ consists of constant functions. If this restriction is degenerate, then, as shown above, the operator D_0 is self-adjoint, and therefore the spaces $\widetilde{H}_0$ and $\widetilde{H}_2$ are orthogonal. In either case the space H_2^+ decomposes into the direct sum of its subspaces $\widetilde{H}_0$ and $\widetilde{H}_2$. Hence there exist functions $g \in \widetilde{H}_0$ and $f_2 \in H_2$ for which $(1 - D_0) f_2 - g = h_0$. Moreover, $D_0 g = g$.

For each $R > 0$ we set $h_R = 1 + iR f_1 + R^2 f_2$. Obviously, there exists a constant $I_3 > 0$ such that $\|h_R - 1\| \leqslant I_3 R$ and $\|h_R\| \leqslant I_3$ for $0 < R \leqslant R_0$. It follows from the relations $(1 - D_0) 1 = 0$, $(1 - D_0) f_1 = D_1 1$, and $(1 - D_0) f_2 - g = D_2 1 - D_1 f_1$ that

$$\|(D_0 + iRD_1 + R^2 D_2) h_R - (1 - gR^2) h_R\| \leqslant I_4 R^3$$

for $0 < R \leqslant R_0$, where I_4 is a positive constant. Hence

$$\|C_R h_R - (1 - gR^2) h_R\| \leqslant (I_2 I_3 + I_4) R^3$$

for $0 < R \leqslant R_0$.

Arbitrary functions $h, \widetilde{h} \in L_2(S^1)$ satisfy the identity

$$\|h+\widetilde{h}\|^2 + \|h-\widetilde{h}\|^2 = 2\|h\|^2 + 2\|\widetilde{h}\|^2.$$

Thus, if $\|h\| = \|\widetilde{h}\|$ and $\|h+\widetilde{h}\| = \|h\|+\|\widetilde{h}\|$, then $h = \widetilde{h}$. This leads to the following conclusion: for each $h \in L_2(S^1)$ the norms of $\|D_0 h\|$ and $\|h\|$ are equal if and only if the functions $U_1 h, \dots, U_k h$ coincide. Hence the equality $D_0 h = h$ is equivalent to the condition that $U_j h = h$ for $j = 1, \dots, k$. Thus, in our case the function g is invariant with respect to the operators $U_1, \dots, U_k$. This means that the operator of multiplication by the function g commutes with the operators $U_1, \dots, U_k$, which, in turn, shows that multiplication by g commutes with all operators C_R, $R > 0$. Moreover, the function g is continuous and real-valued because $g \in H_2^+$.

To complete the proof it remains to show that the function g is everywhere positive. Obviously, $\|C_R\| \leqslant 1$ for each $R > 0$. Since the operator of multiplication by the function g commutes with the operator C_R, it follows that

$$\|C_R^n h_R - (1 - gR^2)C_R^{n-1} h_R\| \leqslant (I_2 I_3 + I_4)R^3,$$

for $0 < R \leqslant R_0$ and each positive integer n, which yields

$$\|C_R^m h_R - (1 - gR^2)^m h_R\| \leqslant I_5 R^3,$$

where $I_5 = m(1 + R_0^2 \sup|g|)^{m-1}(I_2 I_3 + I_4)$. Consider arbitrary $\varepsilon > 0$. We set $E_\varepsilon = \{t \in S^1 \mid g(t) < I_1 - \varepsilon\}$. Let χ_ε be the function on S^1 that is the characteristic function of the set E_ε. For $R \in (0, R_0]$ we have $\|\chi_\varepsilon C_R^m h_R - (1 - gR^2)^m \chi_\varepsilon h_R\| \leqslant I_5 R^3$. Since the function g is invariant with respect to the operators $U_1, \dots, U_k$, so also is χ_ε. Hence the operator C_R commutes with the operator of multiplication by χ_ε. In particular, $\chi_\varepsilon C_R^m h_R = C_R^m(\chi_\varepsilon h_R)$. Then $\|\chi_\varepsilon C_R^m h_R\| \leqslant (1 - I_1 R^2)\|\chi_\varepsilon h_R\|$. Moreover,

$$\|(1 - gR^2)\chi_\varepsilon h_R\| \geqslant (1 - (I_1 - \varepsilon)R^2)\|\chi_\varepsilon h_R\|$$

because $1 - gR^2 \geqslant 1 - (I_1 - \varepsilon)R^2$ on the set E_ε. This yields

$$\varepsilon R^2 \|\chi_\varepsilon h_R\| \leqslant \|(1 - gR^2)\chi_\varepsilon h_R\| - \|\chi_\varepsilon C_R^m h_R\| \leqslant I_5 R^3.$$

Hence $\|\chi_\varepsilon\| \leqslant \|\chi_\varepsilon h_R\| + \|h_R - 1\| \leqslant (\varepsilon^{-1} I_5 + I_3)R$ for $0 < R \leqslant R_0$, which means that $\chi_\varepsilon = 0$ almost everywhere on S^1 and the set E_ε is empty because it is open. Since ε could be chosen arbitrarily small, it follows that $\inf g \geqslant I_1 > 0$.

For $a \in (0,1)$ we define a sequence $p_{0,a}, p_{1,a}, \dots$ of polynomials by the recurrence relations $p_{0,a}(z) = 1$, $p_{1,a}(z) = z$, and $p_{n+1,a}(z) = (1+a)z p_{n,a}(z) - a p_{n-1,a}(z)$ for $n \geqslant 1$.

Lemma 3.10. *The polynomials* $p_{0,a}, p_{1,a}, \dots$ *satisfy conditions* (W1), (W1b), (W2) *formulated in* § 2.

Proof. By [5], Lemma 3.6 the family of polynomials $p_{n,a}$, $n = 1, 2, \dots$, satisfies conditions (W1) and (W2). The addition of the polynomial $p_{0,a}$ to this family does not violate these conditions. We verify condition (W1b). Let $z_a = 2a^{1/2}(1+a)^{-1}$. For $z \in (z_a, 1]$ and each integer $n \geqslant 0$ we set

$$q_n(z) = K_1(z)\lambda_1^n(z) + K_2(z)\lambda_2^n(z),$$

where

$$\lambda_1(z) = (1+a)\frac{z}{2} + \left((1+a)^2\frac{z^2}{4} - a\right)^{1/2},$$

$$\lambda_2(z) = (1+a)\frac{z}{2} - \left((1+a)^2\frac{z^2}{4} - a\right)^{1/2},$$

$$K_1(z) = \frac{z - \lambda_2(z)}{\lambda_1(z) - \lambda_2(z)}, \qquad K_2(z) = \frac{\lambda_1(z) - z}{\lambda_1(z) - \lambda_2(z)}.$$

Moreover, $q_n(z) = p_{n,a}(z)$. In fact, the conditions $q_0(z) = 1$ and $q_1(z) = z$ are ensured by our choice of the functions K_1 and K_2, and the recurrence relations $q_{n+1}(z) = (1+a)zq_n(z) - aq_{n-1}(z)$ follow from the fact that $\lambda_1(z)$ and $\lambda_2(z)$ are the roots of the equation $\lambda^2 = (1+a)z\lambda - a$.

The functions λ_1, λ_2, K_1, and K_2 are infinitely differentiable on the interval $(z_a, 1]$. Direct calculation shows that $\lambda_1(1) = 1$, $\lambda_1'(1) = (1+a)/(1-a) > 0$, $\lambda_2(1) = a$, $K_1(1) = 1$, and $K_2(1) = 0$. Hence there exist constants $I_1 > 0$ and $\varepsilon \in (0, 1 - z_a)$ such that

$$0 < \lambda_2(1-z) < \lambda_1(1-z) \leqslant 1, \qquad 0 < 1 - \lambda_1'(1)z \leqslant 1,$$
$$|\lambda_1(1-z) - 1 + \lambda_1'(1)z| \leqslant I_1 z^2,$$
$$|K_1(1-z) - 1| \leqslant I_1 z, \qquad |K_2(1-z)| \leqslant I_1 z,$$

for $0 \leqslant z \leqslant \varepsilon$. Then for each integer $n \geqslant 0$ one has the estimate

$$|p_{n,a} - \lambda_1^n| \leqslant |K_1 - 1| + |K_2|$$

on $[1-\varepsilon, 1]$. Moreover, $|\lambda_1^n(1-z) - (1-\lambda_1'(1)z)^n| \leqslant n|\lambda_1(1-z) - 1 + \lambda_1'(1)z| \leqslant I_1 n z^2$ for $0 \leqslant z \leqslant \varepsilon$. Thus, $|p_{n,a}(1-z) - (1-\lambda_1'(1)z)^n| \leqslant 2I_1 z + I_1 n z^2$ for $0 \leqslant z \leqslant \varepsilon$. This demonstrates condition (W1b).

Proof of Theorem 1.4. Let $A_1, \dots, A_k$ be isometries of the plane P having no common fixed point and preserving no common lattice of the form $\mathcal{L}_Q$, where Q is a strip, a square, or a regular triangle. We denote by $\mathcal{H}$ the semigroup generated by the isometries $A_1, \dots, A_k$ and set $C = k^{-1}(u[A_1] + \cdots + u[A_k])$. We introduce a Cartesian coordinate system ξ in the plane P. Let $\{C_R\}_{R>0}$ be the family of radial operators corresponding to the operator C in the coordinate system ξ. If the isometries $A_1, \dots, A_k$ satisfy one of conditions (i) and (ii) in Theorem 1.4, then by Lemma 3.6, the action $\mathrm{FSG}_k[A_1, \dots, A_k]$ satisfies condition (2) in the same theorem. Otherwise it follows from Lemmas 3.4 and 3.5 that the semigroup $\mathcal{H}$ either contains a rotation through an angle incommensurable with π or is a group. Then by Lemmas 3.7 and 3.8 the operator family $\{C_R\}_{R>0}$ has properties (V1′) and (V2′). By Lemma 3.9 this family has property (V1b). Further, it follows from Lemma 2.2 that the operator C_R depends continuously on the parameter R. Moreover, $\|C_R\| \leqslant 1$ because C_R is the arithmetic mean of several unitary operators. Thus, the family $\{C_R\}_{R>0}$ satisfies the assumptions of Proposition 2.5. In accordance with this proposition the family of operators C_R^n, $R > 0$, $n = 0, 1, \dots$, has properties (P1), (P1b), and (P2). We observe that the operators C_R^n, $R > 0$, are radial operators corresponding to the operator C^n in the coordinate system ξ. In their turn, the operators $1, C, C^2, \dots$ are averaging operators assigned to the action of $\mathrm{FSG}_k[A_1, \dots, A_k]$. By Proposition 2.3 this action satisfies condition (1) of Theorem 1.4. This completes the proof of Theorem 1.4.

Proof of Theorem 1.3. For arbitrary isometries $A_1, \dots, A_k$ the transformations A_1, A_1^{-1}, ..., A_k, A_k^{-1} satisfy neither of conditions (i) and (ii) in Theorem 1.4. Thus, the assertion of Theorem 1.3 in its part relating to actions of the form

$$\mathrm{FSG}_{2k}[A_1, A_1^{-1}, \dots, A_k, A_k^{-1}]$$

is a consequence of Theorem 1.4. Consider now the action $d = G[A_1, \dots, A_k]$, where $G = \mathrm{FG}_k$ or (under the assumption that A_1, ..., A_k are involutions) $G = \mathbb{Z}_2^{*k}$. We set $a = (2k-1)^{-1}$ if we consider the action of the group FG_k, and $a = (k-1)^{-1}$ if we consider the action of $\mathbb{Z}_2^{*k}$. We note that $k \geqslant 2$, and if $A_1, \dots, A_k$ are involutions, then $k \geqslant 3$ because otherwise the isometries $A_1, \dots, A_k$ have a common fixed point or preserve a lattice generated by a strip. In particular, $0 < a < 1$. Averaging operators assigned to the action d have the form $C^{(n)} = p_{n,a}(D)$, where $D = (2k)^{-1}(u[A_1] + u[A_1^{-1}] + \dots + u[A_k] + u[A_k^{-1}])$ (if $A_1, \dots, A_k$ are involutions, then $D = C = k^{-1}(u[A_1] + \dots + u[A_k])$). In fact, the equalities $C^{(0)} = 1$ and $C^{(1)} = D$ are obvious, and the recurrence relations $C^{(n+1)} = (1+a)DC^{(n)} - aC^{(n-1)}$, $n = 1, 2, \dots$, can be readily verified (cf. the proof of Theorems 1.1 and 1.2 in [5]). By Lemma 2.2, $C_R^{(n)} = p_{n,a}(D_R)$ for each $R > 0$ and each integer $n \geqslant 0$. The isometries $A_1, A_1^{-1}, \dots, A_k, A_k^{-1}$ satisfy neither of the conditions (i) and (ii) of Theorem 1.4, and, of course, the semigroup of isometries generated by these isometries is a group. Then it follows from Lemmas 3.8 and 3.9 that the operator family $\{D_R\}_{R>0}$ has properties (V1′), (V1b), and (V2′). The operator D_R depends continuously on the parameter R and is self-adjoint since $D_R = (C_R + C_R^*)/2$. Moreover, $\|D_R^n\| = \|D_R\|^n$ for each positive integer n. Hence for the family $\{D_R\}_{R>0}$ property (V1′) is equivalent to (V1), and property (V2′) to (V2). Thus, the family $\{D_R\}_{R>0}$ satisfies all the assumptions of Proposition 2.4. The polynomials $p_{0,a}, p_{1,a}, \dots$ have real coefficients and satisfy conditions (W1), (W1b), and (W2) by Lemma 3.10. By Proposition 2.4 the family of radial operators $C_R^{(n)}$, $R > 0$, $n = 0, 1, \dots$, has properties (P1), (P1b), and (P2). Using Proposition 2.3 we complete the proof of Theorem 1.3.

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