On maximal subgroups of ample groups

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Abstract

The paper is concerned with maximal subgroups of the ample (better known as topological full) groups of homeomorphisms of totally disconnected compact metrizable topological spaces. We describe all maximal subgroups that are stabilizers of finite sets. Under certain assumptions on the ample group (including minimality), we describe all maximal subgroups that are stabilizers of closed sets or stabilizers of partitions into clopen sets. In particular, our results apply to the ample groups associated with Cantor minimal systems.

1 Introduction

In this paper we study maximal subgroups of the ample groups (the latter are better known as the topological full groups). Let us begin with an overview of maximal and related to them weakly maximal subgroups.

A proper subgroup H of a group G is *maximal* if there is no group placed between H and G in the lattice of subgroups of G. Maximal subgroups play a very important role in group theory. The task of describing all maximal subgroups of a group G is equivalent to the task of classifying all primitive actions of G (an action of the group on a set X is *primitive* if it preserves no nontrivial equivalence relations on X). Indeed, if H is a maximal subgroup of G then the natural action of G on the cosets of H is primitive. Conversely, if $\alpha : G \curvearrowright X$ is a primitive action on a set X that is not a singleton then the point stabilizers $\mathrm{St}_\alpha(x)$, $x \in X$ are maximal subgroups of the group G. Maximal subgroups can also arise as the stabilizers of sets or collections of sets (e.g., partitions).

The problem of describing all maximal subgroups of a given group attracted a lot of attention. Among the most remarkable results here is the complete solution of this problem for finite symmetric groups (based on the O'Nan-Scott theorem), which was obtained in the 1980s as one of the first applications of the classification of finite simple groups (see, e.g., Section 8.5 in the book [DM]). There is also an understanding on how to approach this problem for general finite groups (as outlined by Aschbacher and Scott [AS]). Much less is known about maximal subgroups of infinite groups. One may be interested whether such a group has a maximal subgroup of infinite index. For countable groups, this question is closely related to the question about *primitivity* of the group, that is, existence of a faithful primitive action. Another question is to determine how many maximal subgroups are there, say, whether a given countable group has a *continuum* (that is, an uncountable set of cardinality $2^{\aleph_0}$) of maximal subgroups. If an infinite group does not have many maximal subgroups, it makes sense to look for *weakly maximal subgroups*, which are subgroups of infinite index maximal with respect to this property. Note that in the case a countable group G is finitely generated, any subgroup of finite index in G is finitely generated as well, which implies that there are at most countably many such subgroups. More importantly, it follows that any proper subgroup of G is contained in a maximal subgroup while any subgroup of infinite index is contained in a weakly maximal subgroup.

A substantial progress in the late 1970s was achieved by Margulis and Soifer [MS1, MS2, MS3]. Answering a question of Platonov, they obtained fundamental results on maximal subgroups of finitely generated linear groups. The first of their results is that any such group admits a maximal subgroup of infinite index if and only if it is not virtually solvable. Another result is that the free nonabelian group F_n on $n \geq 2$ generators has a continuum of maximal subgroups. These results were extended by Gelander and Y. Glasner [GG] to general countable linear groups. Later Gelander and Meiri [GM] showed that each of the groups $SL_n(\mathbb{Z})$, $n \geq 3$ has a continuum of maximal subgroups. For more on these and related topics, see the survey [GGS].

A completely different story is told by the groups of branch type. The class of the *branch groups* was introduced in [Gri3] in relation to the study of just infinite groups (these are infinite groups in which every proper quotient is finite) and as an abstract model behind the family of groups $\mathcal{X} = \{G_\omega\}$, $\omega \in \{0,1,2\}^{\mathbb{N}}$ constructed in [Gri1]. This family mostly consists of groups of intermediate growth (faster than polynomial but slower than exponential). Two notable representatives of the family are the groups $\mathcal{G}_{(012)^\infty}$ (the "first" Grigorchuk group) and $\mathcal{G}_{(01)^\infty}$ (the Grigorchuk-Erschler group) given by periodic sequences $(012)^\infty = 012012\ldots$ and $(01)^\infty = 0101\ldots$. These two groups were also at the root of studies of the class of *self-similar groups* (see the book [Nek1]). Among recent discoveries is the fact that they are related to substitution subshifts generated by primitive substitutions, in particular, the period doubling and Morse subshifts (see [M-B, GLN, GV]). These subshifts are important representatives of the Cantor minimal systems (the latter are important in our study).

The question of Hartly from 1993 about existence of maximal subgroups of infinite index in the group $\mathcal{G}_{(012)^\infty}$ was answered in the negative by Pervova [Per1]. This result made a big impact on the study of $\mathcal{G}_{(012)^\infty}$ and led to establishing such properties as subgroup separability (also called the LERF property) and decidability of generalized word problem (see [GW]). In [Per2], Pervova extended her result to some branch groups in a family that was named the GGS (Grigorchuk-Gupta-Sidki) groups in the book [Bau]. The groups considered by Pervova are torsion groups (that is, all elements are of finite order), and presently there are no known examples of finitely generated torsion branch groups admitting a maximal subgroup of infinite index. Recently Francoeur and Thillaisundaram [FT] extended Pervova's result to all GGS groups, including non-torsion groups. On the other hand, Bondarenko [Bon] constructed an example of a non-torsion branch group with a maximal subgroup of infinite index. Later Francoeur and Garrido [FG] discovered that, in fact, the group $\mathcal{G}_{(01)^\infty}$ has maximal subgroups of infinite index. There are only countably many of those and they are all described. Also, it was shown that the non-torsion iterated monodromy groups of the tent map (a special case of some groups first introduced by Šunić in [Sun] as "siblings of the Grigorchuk group") have exactly countably many maximal subgroups of infinite index (those are described up to conjugacy). Presently there are no known examples of finitely generated branch groups with uncountably many maximal subgroups. Another relevant fact is that the branch groups cannot act quasi-2-transitively on infinite sets (see [Fra2]).

The *weakly branch groups* are a far going generalization of the class of branch groups. Their introduction was initiated by the studies around the Basilica group $\mathcal{B}$ invented in [GZ] as a group generated by a 3-state automaton over the binary alphabet. Bartholdi and Virag [BV] proved amenability of $\mathcal{B}$, which produced the first example of amenable but not subexponentially amenable group (answering the question from [Gri2] about existence of such groups). Later it was discovered that $\mathcal{B}$ is isomorphic to the iterated monodromy group of the polynomial $z^2 - 1$ [BGN, Nek1]. In the paper [Fra1], Francoeur extended techniques of Pervova to a large class of weakly branch groups, which allowed him not only to prove that any maximal subgroup in a branch group is itself branch, but also to prove that the Basilica group has maximal subgroups of infinite index and hence is primitive (as each proper quotient of $\mathcal{B}$ is virtually nilpotent).

The next class of infinite groups whose maximal subgroups attracted attention of researchers are the *Higman-Thompson type groups*. Savchuk [Sav1, Sav2] showed that all orbits of the action of Thompson's group F on the interval $(0,1)$ are primitive, which implies that the point stabilizers $\mathrm{St}_F(x)$, $x \in (0,1)$ are maximal subgroups of infinite index. The associated Schreier coset graphs are quasi-isometric to a tree. It was also shown that $\mathrm{St}_F(x)$ is not finitely generated if x is irrational while being finitely generated for $x \in \mathbb{Z}[1/2]$. This study was extended by Golan and Sapir [GS1, GS2, GS3], Aiello and Nagnibeda [AN1, AN2], and others. Among other results, Golan and Sapir produced the first example of a maximal subgroup of F that does not arise as the stabilizer of a point. Another Thompson's group V is considered by Belk, Bleak, Quick and Skipper in [BBQS] where an uncountable family of maximal subgroups of V is produced. Moreover, it is shown that there are uncountably many pairwise non-isomorphic maximal subgroups of V.

Weakly maximal subgroups of finitely generated infinite groups can often play a role similar to that of maximal subgroups of infinite index. For instance, they are useful in the study of profinite groups (Barnea and Shalev [BS]). As was observed in [BG1, BG2], stabilizers of points on the boundary of the rooted tree on which a weakly branch group acts are weakly maximal subgroups. In the case of branch groups from the family $\mathcal{X}$, the associated Schreier coset graphs have surprisingly simple, linear geometric structure. Almost all of them are quasi-isometric to the Cayley graph of $\mathbb{Z}$. When taking into account edge labels, the structure becomes more complicated but still controlled. For the group $\mathcal{G}_{(012)^\infty}$, for example, the graphs represent the so-called aperiodic order (see [GLN]).

Branch groups, weakly branch groups and groups of Higman-Thompson type are subclasses of the class of *micro-supported groups*. Elements of this class are groups G acting faithfully on a topological space X in such a way that for any nonempty open subset $U \subset X$, the rigid stabilizer $\mathrm{RiSt}_G(U)$ (also referred to as the local subgroup and denoted G_U), which consists of elements acting trivially on $X \setminus U$, is nontrivial. Also, all these groups represent an even wider class of *dynamically defined groups*, the object of a fast developing new area on the border between theory of dynamical systems and group theory. A recent book by Nekrashevych [Nek3] is an excellent source of information on methods and results of this direction of mathematics.

A rich source of dynamically defined groups are groups that will be called in this paper the *ample groups*. The idea of amplification (or saturation) in dynamics and group theory is quite simple. Given a topological space X and a group G of its homeomorphisms, one can enlarge G to a group $\mathsf{F}(G)$ by adding those homeomorphisms of X that act locally as elements of G. We call the group $\mathsf{F}(G)$ the *full amplification* of G. The group G is called *ample* if $\mathsf{F}(G) = G$. Note that $\mathsf{F}(\mathsf{F}(G)) = \mathsf{F}(G)$ so that the group $\mathsf{F}(G)$ is always ample. This idea works best when X is a Cantor set or, more generally, a totally disconnected compact metrizable space. This is because the topology on such a space is generated by clopen (i.e., both closed and open) sets. Clopen sets allow to cut all homeomorphisms in G into pieces, then new homeomorphisms can be constructed, as a jigsaw puzzle, out of those pieces.

In what follows, all topological spaces are assumed to be totally disconnected, compact and metrizable. There are many situations when continuous group actions on such spaces arise so that the above construction can be used. For example, any countable group G acts naturally by permutations on the space A^G, where A is a finite set with more than one element. The actions of G on closed invariant subsets of A^G are now a popular area of studies. The group of automorphisms of an infinite, locally finite tree acts naturally on the boundary of the tree. Any countable group G acts by conjugation on the space $\mathrm{Sub}(G)$ of its subgroups, where the topology on $\mathrm{Sub}(G)$ is induced by the product topology on 2^G. This list can go on.

The notion of an ample group was introduced by Krieger in [Kri]. The groups considered in [Kri] are locally finite (that is, every finitely generated subgroup is finite), and so this notion has seen limited use being applied only to locally finite groups. We would like to extend it to arbitrary groups of homeomorphisms replacing the common notion of a topological full group. The latter originated in the theory of Cantor minimal systems. A *Cantor minimal system* (X, f) consists of a Cantor set X and a minimal homeomorphism $f : X \to X$. In the paper [GW], E. Glasner and Weiss associated to (X, f) two groups, the *full group* $[f]$ and the *finite full group* $[[f]]$ (notation is from respectively [GPS] and [Mat]). The full group $[f]$ consists of all homeomorphisms of X that leave invariant every orbit of f. It is relevant to the study of orbit equivalence. Any map $g \in [f]$ can be given by a formula $g(x) = f^{n(x)}(x)$, $x \in X$ for some function $n : X \to \mathbb{Z}$. Since the minimal homeomorphism f has no periodic points, it follows that the function n is unique and its level sets are closed. The finite full group $[[f]]$ consists of those $g \in [f]$ for which the function n is continuous or, equivalently, takes only finitely many values. In that case, all level sets of n are clopen. Informally, elements of $[f]$ are "pointwise" elements of the cyclic group $\langle f \rangle$ while elements of $[[f]]$ are "piecewise" elements of $\langle f \rangle$. It is easy to observe that $[[f]] = \mathsf{F}(\langle f \rangle)$, the full amplification of the cyclic group generated by f. The group $[[f]]$ was renamed the *topological full group* (TFG) in [GPS]. An important property of amplification is that the ample group $\mathsf{F}(G)$ is countable whenever G is countable. In particular, the TFG $[[f]]$ is always countable whereas the full group $[f]$ is not.

The remarkable result proved by Giordano, Putnam and Skau [GPS] is that the (isomorphism class of) TFG $[[f]]$ is an almost complete invariant of dynamics of a Cantor minimal system (X, f). To be precise, if (X, f) and (Y, g) are two Cantor minimal systems such that the groups $[[f]]$ and $[[g]]$ are isomorphic then the systems are *flip conjugate*, which means that for some homeomorphism $\phi : X \to Y$ we have $\phi f \phi^{-1} = g$ or g^{-1}. As there is a continuum of pairwise non-flip-conjugate Cantor minimal systems (e.g., systems with different entropies), one gets a continuum of pairwise non-isomorphic TFGs. The result is based on two fundamental facts about TFGs. For simplicity, let us assume that $X = Y$. The first fact, established in [GPS], is that if the TFGs $[[f]]$ and $[[g]]$ are isomorphic then any isomorphism between them is implemented by a conjugation in the group Homeo(X) of all homeomorphisms of X. This property of TFGs seems to be characteristic for various kinds of micro-supported groups (see Section 2.2 in the book [Nek3]). The second fact, derived from an older result of Boyle, is that if the ample groups $[[f]]$ and $[[g]]$ are the same then the cyclic groups $\langle f\rangle$ and $\langle g\rangle$ are conjugate in the group $[[f]] = [[g]]$.

The systematic study of group-theoretic properties of the TFG $[[f]]$ associated with a Cantor minimal system (X, f) was initiated by Matui [Mat]. He showed that the commutator subgroup of $[[f]]$ is simple. In the case when f is a minimal subshift over a finite alphabet, the commutator subgroup is finitely generated (but not finitely presented). The group $[[f]]$ admits a nontrivial homomorphism onto $\mathbb{Z}$ and the kernel of that homomorphism is a product of two locally finite subgroups (the so-called factorization property). Further, the group $[[f]]$ has the property LEF (local embeddability into finite groups), see [GM]. Any finite group and the free abelian group of infinite rank embed into $[[f]]$. Every branch group from the family $\mathcal{X}$ mentioned before embeds into some TFG (Matte Bon [M-B]). The remarkable result obtained by Juschenko and Monod [JM] (confirming a conjecture of Grigorchuk and Medynets [GM]) states that any TFG associated with a Cantor minimal system is amenable. Note that these groups are not elementary amenable, which means that they cannot be obtained from finite and abelian groups using certain natural operations that preserve amenability. This is the second known type of amenable but not elementary amenable groups, the first type being the groups of intermediate growth.

The goal of this paper is to initiate the systematic study of maximal subgroups of general ample groups. Our results are mostly reminiscent of the classification of maximal subgroups in finite symmetric groups. Let us recall that all subgroups of a finite symmetric group are split into three classes: (i) *intransitive subgroups* (those that leave invariant a nontrivial subset), (ii) *imprimitive subgroups* (transitive subgroups that leave invariant a nontrivial partition), and (iii) *primitive subgroups* (the remaining ones). Maximal subgroups in the first two classes are easy to describe. Namely, intransitive maximal subgroups are stabilizers of certain subsets while imprimitive maximal subgroups are stabilizers of certain partitions. Primitive maximal subgroups form a number of subclasses described by the O'Nan-Scott theorem.

When dealing with the ample groups, arbitrary subsets and partitions should be replaced by closed subsets and partitions into closed subsets. Similarly, transitivity is replaced with minimality (which is absence of nontrivial closed invariant subsets). An ample group $\mathcal{G} \subset$ Homeo(X) acting minimally on the topological space X is an analog of a finite symmetric group (in the case when X is finite, $\mathcal{G}$ is exactly the symmetric group). So we split all subgroups of the group $\mathcal{G}$ into three classes: (I) subgroups leaving invariant a nontrivial closed set, (II) subgroups acting minimally on X but leaving invariant a nontrivial partition into closed sets, and (III) topologically primitive subgroups (the remaining ones). Furthermore, we distinguish three subclasses in the first class: (I_1) subgroups leaving invariant a nonempty finite set, (I_2) subgroups leaving invariant an infinite, nowhere dense closed set, and (I_3)

subgroups leaving invariant a nontrivial clopen set. For general subgroups, these subclasses need not be disjoint or cover the entire class (I), but as far as the maximal subgroups are concerned, the subclasses (I_1), (I_2) and (I_3) form a partition of (I). In the class (II), we distinguish one subclass (II_0) consisting of subgroups leaving invariant a nontrivial partition into clopen sets (or, equivalently, a finite partition into closed sets).

We proceed to the description of our main results. The first of them provides a characterization of those maximal subgroups of ample groups that are stabilizers of closed sets. Namely, they are exactly maximal subgroups in the class (I).

Theorem 1.1. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a maximal subgroup of $\mathcal{G}$ that does not act minimally on X. Then $H = \mathrm{St}_{\mathcal{G}}(Y)$, the stabilizer of some closed set $Y \subset X$ different from the empty set and X. Moreover, the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal.*

The next theorem already gives a continuum of maximal subgroups in any ample group acting without finite orbits on a Cantor set. All those subgroups belong to the subclass (I_1) of the above classification.

Theorem 1.2. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that has no finite orbits. Suppose Y is a finite nonempty subset of X. Then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if Y is contained in a single orbit of $\mathcal{G}$.*

For a more general and more detailed result on the stabilizers of finite sets, see Theorem 7.12 below. Theorem 1.2 imposes very modest conditions on the ample group $\mathcal{G}$ (Theorem 7.12 imposes no conditions at all). To treat the stabilizers of infinite closed sets, we need stronger assumptions. Namely, $\mathcal{G}$ has to act minimally on X and to possess another property that we call *Property E* (entanglement): for any clopen sets $U_1, U_2 \subset X$ that overlap, the local subgroup (i.e., rigid stabilizer) $\mathcal{G}_{U_1 \cup U_2}$ is generated by its subgroups $\mathcal{G}_{U_1}$ and $\mathcal{G}_{U_2}$.

Theorem 1.3. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on a Cantor set X and has Property E. Suppose $Y \subset X$ is an infinite closed set that is nowhere dense in X. Then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if it acts minimally when restricted to Y.*

Proposition 8.2 below shows that Theorem 1.3 provides uncountably many examples of maximal subgroups of the ample group $\mathcal{G}$ (different from the stabilizers of finite sets).

If an ample group $\mathcal{G}$ acts minimally on a Cantor set X and the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set Y is a maximal subgroup of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally on Y. This implies that the set Y is either finite, or infinite and nowhere dense, or clopen. The first two cases are covered by Theorems 1.2 and 1.3. It remains to consider the stabilizers of clopen sets.

Theorem 1.4. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on a Cantor set X and has Property E. Suppose U is a clopen set different from the empty set and X. Then $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$, the stabilizer of the partition $X = U \sqcup (X \setminus U)$, is a maximal subgroup of $\mathcal{G}$. If U cannot be mapped onto $X \setminus U$ by an element of $\mathcal{G}$ then $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$; otherwise $\mathrm{St}_{\mathcal{G}}(U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$.*

In the case when the clopen set U can be mapped onto $X \setminus U$, the stabilizer $\mathrm{St}_{\mathcal{G}}(U)$ is not a maximal subgroup of $\mathcal{G}$. However, $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ is the only group placed between $\mathrm{St}_{\mathcal{G}}(U)$ and $\mathcal{G}$ in the lattice of subgroups of $\mathcal{G}$.

In addition to the stabilizers of clopen sets, Theorem 1.4 also treats the stabilizers of partitions into two clopen sets. Our next result covers the stabilizers of partitions into three or more clopen sets.

Theorem 1.5. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on a Cantor set X and has Property E. Suppose $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into at least three nonempty clopen sets. Then the stabilizer $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ of the partition is a maximal subgroup of $\mathcal{G}$ if and only if its induced action on the set $\{U_1, U_2, \ldots, U_k\}$ is transitive.*

Our last result on maximal subgroups provides a characterization of those maximal subgroups of ample groups that are the stabilizers of partitions into clopen sets.

Theorem 1.6. *Let X be a Cantor set and $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group with Property E. Suppose H is a maximal subgroup of $\mathcal{G}$ that acts minimally on X and contains a local group $\mathcal{G}_U$ for some nonempty clopen set U. Then $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ for some partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ into nonempty clopen sets. Moreover, the partition is unique, it consists of at least two sets, and the induced action of H on the set $\{U_1, U_2, \ldots, U_k\}$ is transitive.*

Theorems 1.3, 1.4, 1.5 and 1.6 require the ample group to have Property E. To find examples of ample groups with this property, we look for the topological full groups associated with Cantor minimal systems.

Theorem 1.7. *For any minimal homeomorphism f of a Cantor set, the ample group $\mathsf{F}(\langle f \rangle)$ has Property E.*

Another example of an ample group with Property E is Thompson's group V. In its representation as a group of homeomorphisms of a Cantor set X, it is ample and acts minimally on X. Property E can be derived from the fact that the group V is simple (see Section 9 for more details).

It would be interesting to find more examples of the ample groups with Property E. More generally, it would be interesting to find out which of our results extend to other classes of micro-supported groups (in particular, the classes mentioned above).

For the ample groups acting minimally on a Cantor set and having Property E, our results describe all maximal subgroups in the subclasses (I_1), (I_2), (I_3) and (II_0). Presumably, the other maximal subgroups in the class (II) are also stabilizers of partitions into closed sets. In the case of finite symmetric groups, there is a large variety of primitive maximal subgroups. Maximal subgroups of Thompson's group V found in [BBQS] include a continuum of topologically primitive subgroups. As for the ample groups associated with Cantor minimal systems, no interesting examples of such subgroups are found so far. This leads us to formulate the following bold conjecture.

Conjecture 1.8. *Let f be a minimal homeomorphisms of a Cantor set X. Suppose H is a maximal subgroup of the ample group $\mathcal{G} = \mathsf{F}(\langle f \rangle)$. Then exactly one of the following statements holds true.*

(i) H is the stabilizer of a closed set $Y \subset X$ different from X and the empty set. Moreover, the induced action of H on Y is minimal. Moreover, the set Y cannot be mapped onto $X \setminus Y$ by elements of the group $\mathcal{G}$.

(ii) H is the stabilizer of a partition $\mathcal{P} = \{Y_\alpha\}_{\alpha \in \mathcal{A}}$ of X into closed sets different from the partition into points and the trivial partition $\{X\}$. Moreover, the partition $\mathcal{P}$ is unique and the induced action of H on the factor space $X/\mathcal{P}$ is minimal.

(iii) H is a normal subgroup of finite prime index in $\mathcal{G}$. Moreover, H contains the commutator subgroup of $\mathcal{G}$.

The paper is organized as follows. In Section 2 we describe the concept of amplification and define the ample groups. In Section 3 we define some important subgroups of the ample groups and, as part of that, introduce generalized permutations. In Section 4 we define various stabilizers associated to a group of homeomorphisms. In Section 5 we introduce a number of useful properties that groups of homeomorphisms can have. In Section 6 we study generalized 2-cycles, which are a class of generalized permutations. The generalized 2-cycles are used in most constructions in our paper. Section 7 is devoted to the study of maximal subgroups of ample groups. In that section we prove slightly generalized versions of Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and 1.6 (respectively Theorems 7.6, 7.13, 7.18, 7.19, 7.20 and 7.21). In Section 8 we describe a construction of closed, nowhere dense sets and groups of homeomorphisms acting on them that allows to provide a wealth of examples for Theorem 1.3. In Section 9 we first discuss Property E in detail. Then we turn to the study of ample groups associated with Cantor minimal systems, which results in the proof of Theorem 1.7 (Theorem 9.11).

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2 Amplification

Let X be a topological space. We denote by $C(X,X)$ the set of all continuous maps $f : X \to X$. This set is a semigroup (and a monoid) with respect to the composition of maps. The identity function $\mathrm{id}_X : X \to X$ is the identity element. Homeomorphisms of X are invertible elements of $C(X,X)$. The set $\mathrm{Homeo}(X)$ of all homeomorphisms is a group.

We assume that the topological space X is compact, metrizable, and totally disconnected. These conditions imply that the topology is generated by *clopen* (that is, both closed and open) sets. The main example is a Cantor set, but a finite set with the discrete topology is an example as well. If Y is a nonempty closed subset of X, then Y as the topological space with the induced topology also satisfies those conditions.

Proposition 2.1. *Given a subset $S \subset C(X,X)$ and an element $f \in C(X,X)$, the following conditions are equivalent:*

(i) f *is* ***locally an element of*** S*, which means that for any point* $x \in X$ *there exists an open neighborhood* U_x *of* x *and an element* $g_x \in S$ *such that* $f|_{U_x} = g_x|_{U_x}$ *(*f *coincides with* g_x *on* U_x*);*

(ii) f *is* ***piecewise an element of*** S*, which means that there exist clopen sets* $V_1, \dots, V_k$ *forming a partition of* X *and elements* $h_1, h_2, \dots, h_k \in S$ *such that* $f|_{V_i} = h_i|_{V_i}$ *for* $1 \le i \le k$*;*

(iii) there exists a locally constant function $\phi : X \to S$ *such that* $f(x) = (\phi(x))(x)$ *for all* $x \in X$*.*

Proof. Assuming condition (ii) holds, we define a function $\phi : X \to S$ by $\phi(x) = h_i$ for $x \in V_i$, $1 \le i \le k$. Then $f(x) = (\phi(x))(x)$ for all $x \in X$. Since the sets $V_1, V_2, \dots, V_k$ are clopen, the function ϕ is locally constant. Hence (ii) $\Longrightarrow$ (iii).

Next assume condition (iii) holds. Since the function ϕ is locally constant, any point $x \in X$ has an open neighborhood U_x such that ϕ is constant on U_x. Then f coincides with $\phi(x) \in S$ on U_x. Hence (iii) $\Longrightarrow$ (i).

We proceed to the main implication (i) $\Longrightarrow$ (ii). Assume condition (i) holds. Since the topological space X is compact and totally disconnected, we can further assume without loss of generality that the neighborhoods U_x, $x \in X$ are clopen. The sets U_x form an open cover of the compact space X. Hence there are finitely many points $x_1, x_2, \dots, x_k \in X$ such that the sets U_{x_i}, $1 \le i \le k$ form a subcover. Now let $V_1 = U_{x_1}$ and $V_i = U_{x_i} \setminus (U_{x_1} \cup \dots \cup U_{x_{i-1}})$ for $i = 2, 3, \dots, k$. Then the sets $V_1, V_2, \dots, V_k$ form a partition of X. Since the sets U_{x_i}, $1 \le i \le k$ are clopen, it follows that the sets V_i, $1 \le i \le k$ are clopen as well. Let $h_i = g_{x_i}$ for $1 \le i \le k$. Clearly, f coincides with h_i on V_i for $1 \le i \le k$. Thus condition (ii) holds. $\square$

Definition 2.2. Given a set $S \subset C(X,X)$, let $\widehat{\mathsf{F}}(S)$ denote the set of all continuous maps in $C(X,X)$ that are locally elements of S (that is, satisfy the three conditions in Proposition 2.1). Given a set $S \subset \mathrm{Homeo}(X)$, let $\mathsf{F}(S) = \widehat{\mathsf{F}}(S) \cap \mathrm{Homeo}(X)$.

Clearly, $S \subset \widehat{\mathsf{F}}(S)$ for any $S \subset C(X,X)$. If $S \subset \mathrm{Homeo}(X)$, then $S \subset \mathsf{F}(S) \subset \widehat{\mathsf{F}}(S)$.

Lemma 2.3. $\widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S)) = \widehat{\mathsf{F}}(S)$ *for any* $S \subset C(X,X)$*.* $\mathsf{F}(\mathsf{F}(S)) = \mathsf{F}(S)$ *for any* $S \subset \mathrm{Homeo}(X)$*.*

Proof. As the inclusion $\widehat{\mathsf{F}}(S) \subset \widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S))$ is obvious, we need to prove that $\widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S)) \subset \widehat{\mathsf{F}}(S)$. Suppose $f \in \widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S))$ and consider an arbitrary point $x \in X$. Since f is locally an element of $\widehat{\mathsf{F}}(S)$, there exists an open neighborhood U of x and a map $g \in \widehat{\mathsf{F}}(S)$ such that $f|_U = g|_U$. Since g is locally an element of S, there exists an open neighborhood W of x and a map $h \in S$ such that $g|_W = h|_W$. Then f coincides with h on $U \cap W$, which is an open neighborhood of x. We conclude that f is locally an element of S.

In the case $S \subset \mathrm{Homeo}(X)$, the inclusion $\mathsf{F}(S) \subset \mathsf{F}(\mathsf{F}(S))$ is obvious. Since $\mathsf{F}(S) \subset \widehat{\mathsf{F}}(S)$, it follows that $\widehat{\mathsf{F}}(\mathsf{F}(S)) \subset \widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S))$. By the above, $\widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S)) = \widehat{\mathsf{F}}(S)$. Therefore $\mathsf{F}(\mathsf{F}(S)) = \widehat{\mathsf{F}}(\mathsf{F}(S)) \cap \mathrm{Homeo}(X) \subset \widehat{\mathsf{F}}(S) \cap \mathrm{Homeo}(X) = \mathsf{F}(S)$. Thus $\mathsf{F}(\mathsf{F}(S)) = \mathsf{F}(S)$. $\square$

Lemma 2.4. *If* $S \subset C(X,X)$ *is a semigroup, then* $\widehat{\mathsf{F}}(S)$ *is also a semigroup. If* $S \subset \mathrm{Homeo}(X)$ *is a semigroup, then* $\mathsf{F}(S)$ *is also a semigroup.*

Proof. Suppose $S \subset C(X, X)$ is a semigroup and consider any two maps $f, g \in \widehat{\mathsf{F}}(S)$. Given a point $x \in X$, there exists an open neighborhood U of x and a map $\tilde{g} \in S$ such that $g|_U = \tilde{g}|_U$. Further, there exists an open neighborhood W of the point $g(x)$ and a map $\tilde{f} \in S$ such that $f|_W = \tilde{f}|_W$. Then the composition fg coincides with $\tilde{f}\tilde{g}$, which belongs to S, on $g^{-1}(W) \cap U$, which is an open neighborhood of x. Hence fg is locally an element of S. Thus $\widehat{\mathsf{F}}(S)$ is a semigroup.

In the case $S \subset \mathrm{Homeo}(X)$, the set $\mathsf{F}(S)$ is a semigroup as it is the intersection of two semigroups $\widehat{\mathsf{F}}(S)$ and $\mathrm{Homeo}(X)$. □

Lemma 2.5. *If $S \subset \mathrm{Homeo}(X)$ is a group, then $\mathsf{F}(S)$ is also a group.*

Proof. $\mathsf{F}(S)$ is a semigroup due to Lemma 2.4. It remains to show that for any homeomorphism $f \in \mathsf{F}(S)$, the inverse f^{-1} is also in $\mathsf{F}(S)$. Take an arbitrary point $x \in X$. Since f is locally an element of S, there exists an open neighborhood U of the point $f^{-1}(x)$ and a map $g \in S$ such that $f|_U = g|_U$. Then f^{-1} coincides with g^{-1}, which belongs to S, on $f(U)$, which is an open neighborhood of x. Thus f^{-1} is locally an element of S. □

Definition 2.6. Given subgroups G and $\widetilde{G}$ of $\mathrm{Homeo}(X)$, we say that $\widetilde{G}$ **amplifies** G if $G \subset \widetilde{G} \subset \mathsf{F}(G)$. The group $\mathsf{F}(G)$ is called the **full amplification** of G. In the case $\mathsf{F}(G) = G$, the group G is called **ample** or **fully amplified**.

Definition 2.7. Given a group $G \subset \mathrm{Homeo}(X)$, for any $x \in X$ we denote by $\mathrm{Orb}_G(x)$ the **orbit** of the point x under the natural action of the group: $\mathrm{Orb}_G(x) = \{g(x) \mid g \in G\}$. We refer to any set of the form $\mathrm{Orb}_G(x)$ as an orbit of the group G.

Lemma 2.8. *If a group $\widetilde{G}$ amplifies a group G then both groups have the same orbits.*

Proof. If a group $\widetilde{G}$ amplifies a group G, then every element of G is also an element of $\widetilde{G}$ while every element of $\widetilde{G}$ is locally an element of G. It follows that for any points $x, y \in X$ we have $y = g(x)$ for some $g \in G$ if and only if $y = \tilde{g}(x)$ for some $\tilde{g} \in \widetilde{G}$. Consequently, the groups G and $\widetilde{G}$ have the same orbits. □

3 Subgroups of a fully amplified group

Let G be a subgroup of $\mathrm{Homeo}(X)$ and $\mathsf{F}(G)$ be its full amplification.

Definition 3.1. Let $\mathsf{K}(G)$ denote the set of all elements of the group $\mathsf{F}(G)$ that belong to the kernel of every nontrivial homomorphism $\phi : \mathsf{F}(G) \to \mathbb{Z}$. Let $\mathsf{Fin}(G)$ denote the subgroup of $\mathsf{F}(G)$ generated by all elements of finite order.

It is easy to see that $\mathsf{K}(G)$ and $\mathsf{Fin}(G)$ are normal subgroups of $\mathsf{F}(G)$ and $\mathsf{Fin}(G) \subset \mathsf{K}(G)$.

For any integer $n \geq 1$ let S_n denote the *symmetric group* of all permutations on the set $\{1, 2, \ldots, n\}$.

Definition 3.2. Let U be a clopen subset of X. Suppose $f_1, f_2, \dots, f_n$ are homeomorphisms of X such that the images $f_1(U), f_2(U), \dots, f_n(U)$ are disjoint. Then for any permutation $\pi \in \mathsf{S}_n$ we define a **generalized permutation** $\mu[U; f_1, f_2, \dots, f_n; \pi] : X \to X$ by

$$\mu[U; f_1, f_2, \dots, f_n; \pi](x) = \begin{cases} f_{\pi(i)}(f_i^{-1}(x)) & \text{if } x \in f_i(U),\ 1 \le i \le n, \\ x & \text{otherwise.} \end{cases}$$

The generalized permutation $\mu[U; f_1, f_2, \dots, f_n; \pi]$ is a homeomorphism of X that permutes disjoint clopen sets $f_1(U), f_2(U), \dots, f_n(U)$ while fixing the rest of the space. If the sets $f_1(U), f_2(U), \dots, f_n(U)$ are not disjoint then $\mu[U; f_1, f_2, \dots, f_n; \pi]$ is not defined.

Lemma 3.3. *Suppose that a generalized permutation $g = \mu[U; f_1, f_2, \dots, f_n; \pi]$ is defined. Then for any homeomorphism $h : X \to X$ we have $\mu[h^{-1}(U); f_1 h, f_2 h, \dots, f_n h; \pi] = g$ and $\mu[U; hf_1, hf_2, \dots, hf_n; \pi] = hgh^{-1}$ (in particular, both generalized permutations are defined).*

Proof. Since the generalized permutation g is defined, the set U is clopen and its images $f_1(U), f_2(U), \dots, f_n(U)$ are disjoint. Then the set $h^{-1}(U)$ is also clopen. The sets $f_i h(h^{-1}(U))$, $1 \le i \le n$ coincide with the sets $f_i(U)$, $1 \le i \le n$. The sets $hf_i(U)$, $1 \le i \le n$ are their images under the homeomorphism h, hence they are disjoint as well. It follows that generalized permutations $g_1 = \mu[h^{-1}(U); f_1 h, f_2 h, \dots, f_n h; \pi]$ and $g_2 = \mu[U; hf_1, hf_2, \dots, hf_n; \pi]$ are defined.

Take any $x \in X$. If $x \in f_i(U)$ for some i, $1 \le i \le n$, then

$$g_1(x) = (f_{\pi(i)} h)\big((f_i h)^{-1}(x)\big) = (f_{\pi(i)} h h^{-1} f_i^{-1})(x) = f_{\pi(i)} f_i^{-1}(x) = g(x).$$

Otherwise $g_1(x) = x = g(x)$. Therefore $g_1 = g$ everywhere on X. Further, if $x \in hf_i(U)$ for some i, $1 \le i \le n$, then $h^{-1}(x) \in f_i(U)$ and we obtain

$$g_2(x) = (hf_{\pi(i)})\big((hf_i)^{-1}(x)\big) = h(f_{\pi(i)} f_i^{-1})(h^{-1}(x)) = h(g(h^{-1}(x))) = (hgh^{-1})(x).$$

If x does not belong to any of the sets $hf_i(U)$, $1 \le i \le n$, then $h^{-1}(x)$ does not belong to any of the sets $f_i(U)$, $1 \le i \le n$. In this case, $g_2(x) = x$ and $g(h^{-1}(x)) = h^{-1}(x)$. It follows that $(hgh^{-1})(x) = h(h^{-1}(x)) = x = g_2(x)$. Thus $g_2 = hgh^{-1}$ everywhere on X. $\square$

Lemma 3.4. *Suppose $U \subset X$ is a clopen set and $f_1, f_2, \dots, f_n$ are homeomorphisms of X such that the images $f_1(U), f_2(U), \dots, f_n(U)$ are disjoint. Then the map $\Phi : \mathsf{S}_n \to \mathrm{Homeo}(X)$ given by $\Phi(\pi) = \mu[U; f_1, f_2, \dots, f_n; \pi]$ is a group homomorphism.*

Proof. Given any permutations $\pi, \sigma \in \mathsf{S}_n$, let $g_1 = \Phi(\pi)$, $g_2 = \Phi(\sigma)$ and $g = \Phi(\pi\sigma)$. We need to show that $g_1(g_2(x)) = g(x)$ for all $x \in X$. First assume $x \in f_i(U)$ for some i, $1 \le i \le n$. Then $g_2(x) = f_{\sigma(i)}(f_i^{-1}(x))$ and $g(x) = f_{\pi\sigma(i)}(f_i^{-1}(x))$. Besides, the point $g_2(x)$ belongs to $f_{\sigma(i)}(U)$, which implies that $g_1(g_2(x)) = f_{\pi(\sigma(i))}(f_{\sigma(i)}^{-1}(g_2(x)))$. We observe that $f_{\sigma(i)}^{-1}(g_2(x)) = f_i^{-1}(x)$. Hence $g_1(g_2(x)) = f_{\pi(\sigma(i))}(f_i^{-1}(x)) = f_{\pi\sigma(i)}(f_i^{-1}(x)) = g(x)$. In the case when x does not belong to any of the sets $f_i(U)$, $1 \le i \le n$, we have $g_1(x) = g_2(x) = g(x) = x$. In particular, $g_1(g_2(x)) = g(x)$. Thus $g_1 g_2 = g$ everywhere on X. $\square$

If homeomorphisms $f_1, f_2, \ldots, f_n$ belong to the group G, then any generalized permutation of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$ belongs to $\mathsf{F}(G)$ as well as to $\mathsf{Fin}(G)$.

Definition 3.5. The **generalized symmetric group** $\mathsf{S}(G)$ over G is the subgroup of $\mathsf{F}(G)$ generated by all generalized permutations in $\mathsf{F}(G)$.

The group $\mathsf{S}(G)$ was introduced (in a more general context) by Nekrashevych [Nek2].

Lemma 3.6. *For any group $G \subset \mathrm{Homeo}(X)$ the generalized symmetric group $\mathsf{S}(G)$ is generated by all maps of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$.*

Proof. Let P be the set of all generalized permutations of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$. It follows from the definition that any element of P is piecewise an element of the group $\mathsf{F}(G)$. Therefore P is contained in the group $\mathsf{F}(\mathsf{F}(G))$, which coincides with $\mathsf{F}(G)$ due to Lemma 2.3.

To prove the lemma, it is enough to show that any generalized permutation $g \in \mathsf{F}(G)$ is a product of elements of the set P. The map g is represented as $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where maps $f_1, f_2, \ldots, f_n$ need not be in $\mathsf{F}(G)$. Let $\pi = \sigma_1\sigma_2 \ldots \sigma_k$ be the decomposition of the permutation π as a product of disjoint cycles. Lemma 3.4 implies that $g = g_1 g_2 \ldots g_k$, where $g_j = \mu[U; f_1, f_2, \ldots, f_n; \sigma_j]$, $1 \le j \le k$. We are going to show that each g_j belongs to P.

First let us show that $g_j \in \mathsf{F}(G)$. For any $i \in \{1, 2, \ldots, n\}$ we have either $\sigma_j(i) = \pi(i)$ or $\sigma_j(i) = i$. In the former case, g_j coincides with g on the clopen set $f_i(U)$. In the latter case, g_j coincides with the identity map on $f_i(U)$. On the complement of the union of sets $f_i(U)$, $1 \le i \le n$, both g and g_j coincide with the identity map. Hence g_j is piecewise an element of the set $\{\mathrm{id}_X, g\}$. Since $g \in \mathsf{F}(G)$, it follows that $g_j \in \mathsf{F}(\mathsf{F}(G)) = \mathsf{F}(G)$.

Recall that the permutation σ_j is a cycle: $\sigma_j = (i_1\, i_2 \ldots i_m)$, where $i_1, i_2, \ldots, i_m$ are distinct elements of $\{1, 2, \ldots, n\}$. Let $W_s = f_{i_s}(U)$, $1 \le s \le m$. Then $W_1, W_2, \ldots, W_m$ are disjoint clopen sets and g_j coincides with the identity map away from them. We have $g_j(W_s) = W_{s+1}$ for $1 \le s \le m-1$ and $g_j(W_m) = W_1$. It follows by induction that $W_s = g_j^{s-1}(W_1)$ for $1 \le s \le m$. Since the cycle σ_j has order m in the group S_n, Lemma 3.4 implies that $g_j^m = \mathrm{id}_X$. As a consequence, $(g_j^{m-1})^{-1}$ coincides with g_j on W_m (as well as anywhere else on X). All this leads to an alternative representation of g_j as a generalized permutation:

$$g_j = \mu\big[W_1; \mathrm{id}_X, g_j, g_j^2, \ldots, g_j^{m-1}; (1\,2 \ldots m)\big].$$

Since g_j and all its powers belong to the group $\mathsf{F}(G)$, we conclude that $g_j \in P$. $\square$

Lemma 3.7. $\mathsf{S}(G)$ *is a normal subgroup of the group* $\mathsf{F}(G)$.

Proof. By Lemma 3.6, the group $\mathsf{S}(G)$ is generated by generalized permutations of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$. Whenever a map $g = \mu[U; f_1, f_2, \ldots, f_n; \pi]$ is defined, for any $h \in \mathrm{Homeo}(X)$ we have $hgh^{-1} = \mu[U; hf_1, hf_2, \ldots, hf_n; \pi]$ due to Lemma 3.3. If the maps $f_1, f_2, \ldots, f_n$ and h belong to the group $\mathsf{F}(G)$, then the maps $hf_1, hf_2, \ldots, hf_n$ are in $\mathsf{F}(G)$ as well. We conclude that the generating set for $\mathsf{S}(G)$ is closed under conjugation by elements of the group $\mathsf{F}(G)$. It follows that the group $\mathsf{S}(G)$ is itself closed under conjugation by elements of $\mathsf{F}(G)$. In other words, $\mathsf{S}(G)$ is a normal subgroup of $\mathsf{F}(G)$. $\square$

Definition 3.8. Let $f \in \text{Homeo}(X)$. A point $x \in X$ is called a **fixed point** of f if $f(x) = x$. The set of all fixed points of f is denoted $\text{Fix}(f)$. Further, a point $x \in X$ is called a **support point** of f if the homeomorphism f does not coincide with the identity map in any neighborhood of x. The set of all support points of f is called the **support** of f and denoted $\text{supp}(f)$.

By definition, the support of f is the complement of the interior of $\text{Fix}(f)$. Equivalently, $\text{supp}(f) = \overline{X \setminus \text{Fix}(f)}$, the closure of the complement of $\text{Fix}(f)$.

Lemma 3.9. *If the supports* $\text{supp}(f)$ *and* $\text{supp}(g)$ *do not share an interior point, then the homeomorphisms f and g commute: $fg = gf$.*

Proof. The interior of $\text{supp}(h)$ is the complement of $\text{Fix}(h)$ for any $h \in \text{Homeo}(X)$. Therefore the assumption of the lemma means that $f(x) = x$ or $g(x) = x$ for any $x \in X$. If $f(x) = g(x) = x$ then, clearly, $f(g(x)) = x = g(f(x))$. In the case when $f(x) = x$ while $g(x) \neq x$, we have $g(g(x)) \neq g(x)$ since g is one-to-one. It follows that $f(g(x)) = g(x) = g(f(x))$. Similarly, if x is not a fixed point of f then neither is $f(x)$. Consequently, both are fixed points of g so that $g(f(x)) = f(x) = f(g(x))$. Thus $fg = gf$ everywhere on X. $\square$

Definition 3.10. Let U be a clopen subset of X. The **local group** (or **local subgroup**) of the ample group $\mathsf{F}(G)$ associated to U, denoted $\mathsf{F}_U(G)$, consists of all maps $f \in \mathsf{F}(G)$ such that $\text{supp}(f) \subset U$. In the case the group G is already ample, we may use alternative notation G_U.

For any clopen set $U \subset X$ and homeomorphism $f : X \to X$, the condition $\text{supp}(f) \subset U$ is equivalent to the condition $f(x) = x$ for all $x \notin U$. It easily follows that the local group $\mathsf{F}_U(G)$ is indeed a subgroup of $\mathsf{F}(G)$. Moreover, $\mathsf{F}_U(G)$ is an ample group.

Lemma 3.11. *If $H = \mathsf{F}_U(G)$ then $gHg^{-1} = \mathsf{F}_{g(U)}(G)$ for any $g \in \mathsf{F}(G)$.*

Proof. The local group H consists of those elements of $\mathsf{F}(G)$ that fix all points not in U. Take any $g \in \mathsf{F}(G)$. A homeomorphism $f \in \text{Homeo}(X)$ fixes a point $x \in X$ if and only if gfg^{-1} fixes the point $g(x)$. Also, $f \in \mathsf{F}(G)$ if and only if $gfg^{-1} \in \mathsf{F}(G)$. It follows that the conjugate subgroup gHg^{-1} consists of those elements of $\mathsf{F}(G)$ that fix all points in $g(X \setminus U)$. Since $g(X \setminus U) = X \setminus g(U)$ and $g(U)$ is a clopen set, we conclude that $gHg^{-1} = \mathsf{F}_{g(U)}(G)$. $\square$

4 Stabilizers

Any group $G \subset \text{Homeo}(X)$ acts naturally on the topological space X. This action induces several other actions. In this section we define various stabilizers related to those actions. Note that the stabilizers of any group action on any set are subgroups of the acting group.

Definition 4.1. The **stabilizer** $\text{St}_G(x)$ of a point $x \in X$ under the action of the group G consists of those elements of G that fix x: $\text{St}_G(x) = \{g \in G \mid g(x) = x\}$. The **neighborhood stabilizer** $\text{St}^\circ_G(x)$ of x consists of those elements of $\text{St}_G(x)$ that coincide with the identity map in a sufficiently small neighborhood of the point x.

The neighborhood stabilizer can be interpreted as the usual stabilizer of a certain action on germs. Let $\sim$ be a relation on the set $\mathrm{Homeo}(X) \times X$ such that $(f, x) \sim (g, y)$ if and only if $x = y$ and the map f coincides with g in a sufficiently small neighborhood of x. Then $\sim$ is an equivalence relation. The equivalence class of a pair (f, x) is called the *germ* of the map f at the point x. Combining the right adjoint action of the group G on $\mathrm{Homeo}(X)$ with the natural action on X, we obtain an action on $\mathrm{Homeo}(X) \times X$ given by $g(f, x) = (fg^{-1}, g(x))$. This action preserves the relation $\sim$, hence it induces an action on the germs. Relative to the latter action, $\mathrm{St}_G^\circ(x)$ is the stabilizer of the germ of id_X (as well as any other homeomorphism) at x.

Lemma 4.2. *The neighborhood stabilizer* $\mathrm{St}_G^\circ(x)$ *is a normal subgroup of* $\mathrm{St}_G(x)$.

Proof. Consider the induced action of the group G on germs described above. Every element of the stabilizer $\mathrm{St}_G(x)$ preserves the set of all germs at the point x. Hence the group $\mathrm{St}_G(x)$ acts on this set. It is easy to observe that $\mathrm{St}_G^\circ(x)$ is the kernel of that action. Therefore it is a normal subgroup of $\mathrm{St}_G(x)$. $\square$

The action of the group G on the set X induces an action on subsets of X.

Definition 4.3. The **(set) stabilizer** $\mathrm{St}_G(Y)$ of a set $Y \subset X$ under the action of the group G consists of those elements of G that map Y onto itself. The **pointwise stabilizer** $\mathrm{St}_G^\star(Y)$ of Y consists of those elements of $\mathrm{St}_G(Y)$ that fix every point of the set Y. The **rigid stabilizer** $\mathrm{RiSt}_G(Y)$ of Y consists of those elements of $\mathrm{St}_G(Y)$ that fix every point not in Y.

The pointwise stabilizer can be interpreted as the usual stabilizer of the induced action of G on ordered subsets of X. Indeed, $\mathrm{St}_G^\star(Y)$ is the stabilizer of the set Y endowed with any linear order.

Note that the local group $\mathsf{F}_U(G)$ of the ample group $\mathsf{F}(G)$ associated to a clopen set $U \subset X$ coincides with the rigid stabilizer $\mathrm{RiSt}_{\mathsf{F}(G)}(U)$.

Lemma 4.4. *The pointwise stabilizer* $\mathrm{St}_G^\star(Y)$ *and the rigid stabilizer* $\mathrm{RiSt}_G(Y)$ *are normal subgroups of* $\mathrm{St}_G(Y)$.

Proof. The set stabilizer $\mathrm{St}_G(Y)$ acts naturally on the set Y. The pointwise stabilizer $\mathrm{St}_G^\star(Y)$ is the kernel of that action. Therefore it is a normal subgroup of $\mathrm{St}_G(Y)$.

For any homeomorphism $f : X \to X$ we have $f(Y) = Y$ if and only if $f(X \setminus Y) = X \setminus Y$. Hence $\mathrm{St}_G(Y) = \mathrm{St}_G(X \setminus Y)$. It follows that $\mathrm{RiSt}_G(Y) = \mathrm{St}_G^\star(X \setminus Y)$. By the above, $\mathrm{RiSt}_G(Y)$ is a normal subgroup of $\mathrm{St}_G(X \setminus Y) = \mathrm{St}_G(Y)$. $\square$

The action of the group G on subsets of X induces an action on collections of such subsets (that is, on the set of subsets of the set of subsets of X).

Definition 4.5. The **(collective) stabilizer** $\mathrm{St}_G(Y_1, Y_2, \dots, Y_k)$ of a collection of distinct sets $Y_i \subset X$, $1 \le i \le k$ under the action of the group G consists of those elements of G that map each set in the collection onto (the same or another) set in the collection. The **individual stabilizer** $\mathrm{St}_G^\bullet(Y_1, Y_2, \dots, Y_k)$ of the collection consists of those elements of $\mathrm{St}_G(Y_1, Y_2, \dots, Y_k)$ that map each set Y_i, $1 \le i \le k$ onto itself.

Clearly, the individual stabilizer $\mathrm{St}^\bullet_G(Y_1, Y_2, \ldots, Y_k)$ is the intersection of the set stabilizers $\mathrm{St}_G(Y_i)$, $1 \le i \le k$. In the case when the collection consists of a single set Y, the notation $\mathrm{St}_G(Y)$ is not ambiguous since the collective stabilizer of the collection $\{Y\}$ coincides with the set stabilizer of the set Y.

Lemma 4.6. *The individual stabilizer* $\mathrm{St}^\bullet_G(Y_1, Y_2, \ldots, Y_k)$ *is a normal subgroup of the group* $\mathrm{St}_G(Y_1, Y_2, \ldots, Y_k)$*.*

Proof. The collective stabilizer $\mathrm{St}_G(Y_1, Y_2, \ldots, Y_k)$ acts naturally on the set $\{Y_1, Y_2, \ldots, Y_k\}$. The individual stabilizer $\mathrm{St}^\bullet_G(Y_1, Y_2, \ldots, Y_k)$ is the kernel of that action. Therefore it is a normal subgroup of $\mathrm{St}_G(Y_1, Y_2, \ldots, Y_k)$. □

5 Regularity properties

In this section we formulate several useful properties that a subgroup G of $\mathrm{Homeo}(X)$ may or may not have. When proving statements on the ample group $\mathsf{F}(G)$, we will usually need some combination of those properties.

Property C (countability)**.** The group G is at most countable.

Lemma 5.1. *If a group* $G \subset \mathrm{Homeo}(X)$ *is at most countable, then its full amplification* $\mathsf{F}(G)$ *is also at most countable.*

Proof. Any element of the set $\widehat{\mathsf{F}}(G)$ is uniquely determined by choosing a finite partition of X into clopen sets and assigning an element of G to every set in the partition. Recall that the topological space X is assumed to be compact and metrizable. Hence the topology admits a finite or countable base. Since any clopen set is compact, it is a union of finitely many elements of the base. If a base is at most countable, then it has at most countably many finite subsets. It follows that we have at most countably many clopen sets. Iterating this argument, we obtain that there are at most countably many finite collections of clopen sets. In particular, there are at most countably many finite partitions of X into clopen sets. If the group G is at most countable, then any Cartesian power G^k is also at most countable. Therefore for any finite partition of X, there are at most countably many ways to assign elements of G to elements of the partition. We conclude that the set $\widehat{\mathsf{F}}(G)$ is at most countable. Then its subset $\mathsf{F}(G)$ is also at most countable. □

Property FrA (free action)**.** The action of the group G on X is *free*, which means that all point stabilizers are trivial: $\mathrm{St}_G(x) = \{\mathrm{id}_X\}$ for all $x \in X$.

Lemma 5.2. *Suppose that the topological space* X *contains at least* 3 *points. Then for any nontrivial group* $G \subset \mathrm{Homeo}(X)$ *its full amplification* $\mathsf{F}(G)$ *cannot have Property FrA.*

Proof. Given a nontrivial group $G \subset \mathrm{Homeo}(X)$, we take any element $g \in G$ different from the identity map and then any point $x \in X$ such that $g(x) \neq x$. Since X contains at least 3 points, there is a point $y \in X$ different from x and $g(x)$. Since the topological space X is totally disconnected, the point x has a clopen neighborhood U_1 that does not contain $g(x)$

and another clopen neighborhood U_2 that does not contain y. Also, the point $g(x)$ has a clopen neighborhood U_3 that does not contain y. Then $U = U_1 \cap U_2$ is a clopen neighborhood of x and $V = U_3 \setminus U_1$ is a clopen neighborhood of $g(x)$ disjoint from U. Moreover, neither U nor V contains the point y. Let $W = U \cap g^{-1}(V)$. Then W is a clopen set containing x. Since $W \subset U$ and $g(W) \subset V$, it follows that the sets W and $g(W)$ are disjoint and neither contains y. In particular, a generalized permutation $f = \mu[W; \mathrm{id}_X, g; (1\,2)]$ is defined. Since $\mathrm{id}_X, g \in G$, the map f belongs to $\mathsf{F}(G)$. By construction, $f(y) = y$ while $f(x) = g(x) \neq x$. Thus the action of the group $\mathsf{F}(G)$ on X is not free. $\square$

Property NSP (no singular points)**.** All point stabilizers of the action of the group G on X coincide with the corresponding neighborhood stabilizers: $\mathrm{St}^\circ_G(x) = \mathrm{St}_G(x)$ for all $x \in X$.

A point $x \in X$ is called a *singular point* of the action of G on X if $\mathrm{St}^\circ_G(x)$ is a proper subgroup of $\mathrm{St}_G(x)$. Property NSP means that the action has no singular points (such actions are sometimes called *regular*). Obviously, Property FrA implies Property NSP.

Lemma 5.3. *If a group $G \subset \mathrm{Homeo}(X)$ has Property NSP, then its full amplification $\mathsf{F}(G)$ also has Property NSP.*

Proof. Take any $f \in \mathsf{F}(G)$ and $x \in X$ such that $f(x) = x$. To establish Property NSP for the group $\mathsf{F}(G)$, we need to show that the map f fixes all points in an open neighborhood of x. Since f is locally an element of G, there exists an open neighborhood U of x and a map $g \in G$ such that $f|_U = g|_U$. In particular, $g(x) = f(x) = x$. Since the group G is assumed to have Property NSP, the map g should fix all points in some open neighborhood W of x. Then f fixes all points in $U \cap W$, which is yet another open neighborhood of x. $\square$

Proposition 5.4. *If a group $G \subset \mathrm{Homeo}(X)$ has Property NSP, then $\mathsf{S}(G) = \mathsf{Fin}(G)$.*

Proof. The group $\mathsf{Fin}(G)$ is generated by all elements of finite order in $\mathsf{F}(G)$. The group $\mathsf{S}(G)$ is generated by all generalized permutations in $\mathsf{F}(G)$, which are elements of finite order. Hence $\mathsf{S}(G) \subset \mathsf{Fin}(G)$. To prove the equality $\mathsf{S}(G) = \mathsf{Fin}(G)$, it is enough to show that any element of finite order in $\mathsf{F}(G)$ is a product of generalized permutations.

Let $f \in \mathsf{F}(G)$ be an element of finite order n. Take any point $x \in X$. Since $f^n(x) = x$, it follows that the sequence $x, f(x), f^2(x), \dots$ is periodic and its period is a divisor of n. Let $d(x)$ denote that period. Then the points $x, f(x), \dots, f^{d(x)-1}(x)$ are distinct. Since the topological space X is totally disconnected, there exist disjoint clopen sets $U_0, U_1, \dots, U_{d(x)-1}$ such that $f^i(x) \in U_i$ for any i, $0 \le i \le d(x) - 1$. Further, the group $\mathsf{F}(G)$ has Property NSP (this follows from Lemma 5.3 since the group G has Property NSP). As $f^{d(x)} \in \mathsf{F}(G)$ and $f^{d(x)}(x) = x$, we obtain that the map $f^{d(x)}$ coincides with the identity map in an open neighborhood U of the point x. Without loss of generality, we may assume U to be clopen. Now let $V_x = U \cap U_0 \cap f^{-1}(U_1) \cap \dots \cap f^{1-d(x)}(U_{d(x)-1})$. By construction, V_x is a clopen neighborhood of x. The sets $V_x, f(V_x), \dots, f^{d(x)-1}(V_x)$ are disjoint from one another since $f^i(V_x) \subset U_i$, $0 \le i \le d(x) - 1$. Besides, the map $f^{d(x)}$ coincides with the identity map on V_x as $V_x \subset U$.

For any divisor k of the number n, let $X^{(k)} = \{x \in X \mid d(x) = k\}$. Then X is the disjoint union of all sets of the form $X^{(k)}$. For any point $x \in X$ we have constructed a clopen neighborhood V_x such that the sets $V_x, f(V_x), \dots, f^{d(x)-1}(V_x)$ are disjoint and $f^{d(x)}(y) = y$ for all $y \in V_x$. It follows that $V_x \subset X^{(d(x))}$. We conclude that all sets of the form $X^{(k)}$ are open. Since X is the disjoint union of them, it follows that all such sets are also closed.

For any divisor k of n, let $\mathcal{W}^{(k)}$ denote the collection of all clopen sets $W \subset X^{(k)}$ with the following property: there exists a clopen set V such that $W = V \sqcup f(V) \sqcup \dots \sqcup f^{k-1}(V)$. Note that $f(f^{k-1}(V)) = V$ since f^k coincides with the identity map on $X^{(k)}$. It follows that $f(W) = W$ for any set $W \in \mathcal{W}^{(k)}$. Our goal is to show that $X^{(k)} \in \mathcal{W}^{(k)}$. First we are going to demonstrate that the collection $\mathcal{W}^{(k)}$ is closed under taking finite unions. It is enough to show that for any two sets $W, W' \in \mathcal{W}^{(k)}$ their union $W \cup W'$ is also in $\mathcal{W}^{(k)}$. We have clopen sets V and V' such that the sets $V, f(V), \dots, f^{k-1}(V)$ partition W while the sets $V', f(V'), \dots, f^{k-1}(V')$ partition W'. Since $f(W) = W$, it follows that $f^i(V' \setminus W) = f^i(V') \setminus W$ for any i, $1 \le i \le k-1$. Therefore the set $W' \setminus W$ is the disjoint union of sets $V' \setminus W, f(V' \setminus W), \dots, f^{k-1}(V' \setminus W)$. Consequently, $W \cup W' = W \sqcup (W' \setminus W)$ is the disjoint union of sets $V, f(V), \dots, f^{k-1}(V), V' \setminus W, f(V' \setminus W), \dots, f^{k-1}(V' \setminus W)$. Consider the set $V'' = V \cup (V' \setminus W)$, which is clopen. Since $f^i(V'') = f^i(V) \cup f^i(V' \setminus W)$ for any i, $1 \le i \le k-1$, we obtain that the sets $V'', f(V''), \dots, f^{k-1}(V'')$ partition $W \cup W'$. Thus $W \cup W' \in \mathcal{W}^{(k)}$.

Take any point $x \in X^{(k)}$. Since the set V_x is clopen, so are its images $f^i(V_x)$, $1 \le i \le k-1$ and the union $W_x = V_x \cup f(V_x) \cup \dots \cup f^{k-1}(V_x)$. Since the sets $V_x, f(V_k), \dots, f^{k-1}(V_x)$ are disjoint and f^k coincides with the identity map on V_x, it follows that $W_x \subset X^{(k)}$ and hence $W_x \in \mathcal{W}^{(k)}$. Note that $x \in V_x \subset W_x$. We obtain that each point of $X^{(k)}$ belongs to a set in the collection $\mathcal{W}^{(k)}$. As every element of the collection is a subset of $X^{(k)}$, this means that the union of all sets in $\mathcal{W}^{(k)}$ is $X^{(k)}$. Since all sets in $\mathcal{W}^{(k)}$ are clopen and $X^{(k)}$ is also clopen (and hence compact), it follows that $X^{(k)}$ can be represented as the union of finitely many sets from $\mathcal{W}^{(k)}$. By the above the collection $\mathcal{W}^{(k)}$ is closed under taking finite unions. As a result, $X^{(k)} \in \mathcal{W}^{(k)}$. Hence there exists a clopen set $V^{(k)}$ such that $X^{(k)} = V^{(k)} \sqcup f(V^{(k)}) \sqcup \dots \sqcup f^{k-1}(V^{(k)})$. Consider a generalized permutation $g^{(k)} = \mu[V^{(k)}; \mathrm{id}_X, f, \dots, f^{k-1}; (1\,2 \dots k)]$. This is a well defined element of $\mathsf{F}(G)$. The support of $g^{(k)}$ is $X^{(k)}$. For any point $y \in f^i(V^{(k)})$, where $0 \le i \le k-2$, we have $g^{(k)}(y) = f^{i+1}(f^i)^{-1}(y) = f(y)$. If $y \in f^{k-1}(V^{(k)})$ then $g^{(k)}(y) = (f^{k-1})^{-1}(y) = f^{1-k}(y)$, which equals $f(y)$ since $f^k(y) = y$. We conclude that $g^{(k)}$ coincides with f everywhere on $X^{(k)}$.

For any divisor k of n, the order of the map f, we have constructed a generalized permutation $g^{(k)} \in \mathsf{F}(G)$ that coincides with f on the set $X^{(k)}$ and coincides with the identity map on $X \setminus X^{(k)}$. Since X is the disjoint union of all sets of the form $X^{(k)}$ and there are only finitely many such sets, it follows that the corresponding generalized permutations $g^{(k)}$ commute with one another (due to Lemma 3.9) and their product equals f. $\square$

Property UR (universal recurrence)**.** For any open set $U \subset X$, any point $x \in U$ and any element $g \in G$ there exists an integer $k \ge 1$ such that $g^k(x) \in U$.

A point $x \in X$ is called *recurrent* relative to a homeomorphism $f : X \to X$ if the sequence $f(x), f^2(x), f^3(x), \dots$ visits every neighborhood of x. Property UR means that every point of X is recurrent relative to every element of the group G.

Property NC (no contraction)**.** If $g(U) \subset U$ for some element $g \in G$ and open set $U \subset X$ then, in fact, $g(U) = U$.

Open sets could be replaced by closed sets in the formulation of Property NC. Indeed, a homeomorphism $f \in \mathrm{Homeo}(X)$ contracts an open set $U \subset X$ (that is, maps it onto a proper subset of U) if and only if the inverse map f^{-1} contracts the closed set $f(X \setminus U)$.

Lemma 5.5. *Property NC is equivalent to Property UR.*

Proof. Assume that the group $G \subset \mathrm{Homeo}(X)$ has Property NC. Take any open set $U \subset X$ and element $g \in G$. Since g is a homeomorphism, the sets $g^{-1}(U), g^{-2}(U), \dots$ are open as well, and so is the union $W = U \cup g^{-1}(U) \cup g^{-2}(U) \cup \dots$. Note that $g^{-1}(W) = g^{-1}(U) \cup g^{-2}(U) \cup g^{-3}(U) \cup \dots$, which implies that $g^{-1}(W) \subset W$. Since $g^{-1} \in G$ and the group G has Property NC, we have $g^{-1}(W) = W$. As a consequence, any point $x \in U$ belongs to $g^{-k}(U)$ for some $k \geq 1$. Then $g^k(x) \in U$. Thus G has Property UR.

Conversely, assume that the group G does not have Property NC. Then there exist an open set $U \subset X$ and an element $g \in G$ such that $g(U)$ is a proper subset of U. The inverse map $h = g^{-1}$ is also an element of G. Take any point $x \in U$ not in $g(U)$. The condition $g(U) \subset U$ implies that $h(X \setminus g(U)) \subset X \setminus U$ and $h(X \setminus U) \subset X \setminus U$. It follows that the sequence $h(x), h^2(x), h^3(x), \dots$ never visits U. We conclude that the group G does not have Property UR. $\square$

Lemma 5.6. *Suppose that the group G does not have Property NC. Then there exist an open set $U \subset X$ and an element $g \in G$ such that $g(U) \subset U$ and the set difference $U \setminus g(U)$ consists of a single point.*

Proof. Lemma 5.5 implies that the group G does not have Property UR. Hence there exist an open set $W \subset X$, a point $x \in W$ and a map $h \in G$ such that the sequence $h(x), h^2(x), h^3(x), \dots$ never visits W. The inverse map $g = h^{-1}$ is also an element of G. Let $S = \{h^k(x) \mid k \geq 1\}$ and $U = X \setminus \overline{S}$, the complement of the closure of S. Since g is a homeomorphism, it follows that $g(\overline{S}) = \overline{g(S)}$. It is easy to see that $g(S) = S \cup \{x\}$. Hence $\overline{g(S)} = \overline{S} \cup \{x\}$. Consequently, $g(U) = X \setminus g(\overline{S}) = X \setminus (\overline{S} \cup \{x\}) = U \setminus \{x\}$. By construction, U is an open set. Besides, $W \subset U$ since W is an open set disjoint from S. In particular, $x \in U$. Thus $g(U) \subset U$ and $U \setminus g(U)$ consists of a single point. $\square$

Property NC– (no contraction of clopen sets)**.** If $g(U) \subset U$ for some element $g \in G$ and clopen set $U \subset X$ then, in fact, $g(U) = U$.

Note that replacing open sets with clopen sets in the formulation of Property UR, we obtain an equivalent property. On the other hand, Lemma 5.6 suggests that Property NC– is much weaker than Property NC.

Property IM (invariant measure)**.** There exists a Borel probability measure ν on X invariant under the action of the group G, which means that $\nu(g(E)) = \nu(E)$ for any $g \in G$ and Borel set $E \subset X$.

Lemma 5.7. *Any Borel probability measure on X invariant under the action of a group $G \subset \mathrm{Homeo}(X)$ is also invariant under the action of its full amplification $\mathsf{F}(G)$.*

Proof. Let ν be a Borel probability measure on X. Take any $f \in \mathsf{F}(G)$. Since f is piecewise an element of G, there exist clopen sets $U_1, U_2, \dots, U_k$ and maps $g_1, g_2, \dots, g_k \in G$ such that $U = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ and f coincides with g_i on U_i, $1 \le i \le k$. Then any Borel set $E \subset X$ is partitioned by Borel sets $E \cap U_i$, $1 \le i \le k$ while its image $f(E)$ is partitioned by Borel sets $f(E \cap U_i)$, $1 \le i \le k$. Note that $f(E \cap U_i) = g_i(E \cap U_i)$ for any i, $1 \le i \le k$ since $f|_{U_i} = g_i|_{U_i}$. We obtain that

$$\nu(E) = \nu(E \cap U_1) + \nu(E \cap U_2) + \dots + \nu(E \cap U_k),$$

$$\nu(f(E)) = \nu(g_1(E \cap U_1)) + \nu(g_2(E \cap U_2)) + \dots + \nu(g_k(E \cap U_k)).$$

Assuming the measure ν is invariant under the action of the group G, we have $\nu(g_i(E \cap U_i)) = \nu(E \cap U_i)$ for $1 \le i \le k$. It follows that $\nu(f(E)) = \nu(E)$. Thus ν is invariant under the action of the group $\mathsf{F}(G)$ as well. $\square$

Property IM+ (fully supported invariant measure)**.** There exists a Borel probability measure ν on X invariant under the action of the group G and, moreover, supported on the entire space X, which means that $\nu(U) > 0$ for any nonempty open set $U \subset X$.

Lemma 5.8. *Property IM+ implies Property NC–.*

Proof. Suppose $g(U) \subset U$, where $U \subset X$ is a clopen set and g is an element of a group $G \subset \mathrm{Homeo}(X)$. Since g is a homeomorphism, the image $g(U)$ is also clopen. Then the set $W = U \setminus g(U)$ is clopen as well. Since U is the disjoint union of sets $g(U)$ and W, we have $\nu(U) = \nu(g(U)) + \nu(W)$ for any Borel probability measure ν on X. If the measure ν is invariant under the action of the group G then $\nu(g(U)) = \nu(U)$, which implies that $\nu(W) = 0$. If, additionally, ν is supported on the entire space X, it further follows that the open set W is empty. Hence $g(U) = U$. Thus Property IM+ implies Property NC–. $\square$

Property M (minimality)**.** The natural action of the group G on X is *minimal*, which means that it admits no closed invariant sets other than the empty set and X itself.

A set $Y \subset X$ is *invariant* under a homeomorphism $f : X \to X$ if $f(Y) \subset Y$. It is invariant under the natural action of a group $G \subset \mathrm{Homeo}(X)$ if $g(Y) \subset Y$ for all $g \in G$. For any $g \in G$, the inverse map g^{-1} is in G as well. Assuming invariance of the set Y, we have $g(Y) \subset Y$ and $g^{-1}(Y) \subset Y$, which implies that, in fact, $g(Y) = Y$ for all $g \in G$.

Lemma 5.9. *The natural action of a group $G \subset \mathrm{Homeo}(X)$ on X is minimal if and only if every orbit of G is dense in X.*

Proof. Any orbit $\mathrm{Orb}_G(x)$ of the group G is clearly invariant under the action of G on X. Since $f(\overline{S}) = \overline{f(S)}$ for any set $S \subset X$ and homeomorphism $f : X \to X$, it follows that the closure $\overline{\mathrm{Orb}_G(x)}$ is also invariant under the action of G. Assuming the action is minimal, we obtain that $\overline{\mathrm{Orb}_G(x)} = X$, that is, the orbit $\mathrm{Orb}_G(x)$ is dense in X.

Conversely, assume that the action of the group G on X is not minimal. Then there exists a closed invariant set $Y \subset X$ different from the empty set and from X. Take any point $y \in Y$. Since Y is invariant, the orbit $\mathrm{Orb}_G(y)$ is contained in Y. Since Y is closed, the closure $\overline{\mathrm{Orb}_G(y)}$ is contained in Y as well. Therefore $\overline{\mathrm{Orb}_G(y)} \neq X$, that is, the orbit $\mathrm{Orb}_G(y)$ is not dense in X. $\square$

Lemma 5.10. *Suppose a topological space X is totally disconnected, compact and metrizable. If there is a group $G \subset \mathrm{Homeo}(X)$ that acts minimally on X, then X is either finite or a Cantor set.*

Proof. If a point $x \in X$ is isolated, then the set $\{x\}$ is clopen. It follows that the set Y of all non-isolated points of X is closed. Any homeomorphism of X maps an isolated point to an isolated point. Hence the set Y is invariant under the action of any group $G \subset \mathrm{Homeo}(X)$. If the action is minimal, then either Y is empty or $Y = X$. In the case $Y = X$, there are no isolated points. Then X is a Cantor set due to Brouwer's characterization of Cantor sets. In the case Y is empty, every point of X is isolated. Hence the partition of X into points is a partition into clopen sets. Then compactness of X implies that it is finite. $\square$

Lemma 5.11. *If a group $G \subset \mathrm{Homeo}(X)$ acts minimally on X then any Borel probability measure on X invariant under the action of G is supported on the entire space X. In particular, Properties IM and M imply Property IM+.*

Proof. Suppose a group $G \subset \mathrm{Homeo}(X)$ acts minimally on X. Take any nonempty open set $U \subset X$. For every $x \in X$, the orbit $\mathrm{Orb}_G(x)$ is dense in X (due to Lemma 5.9). Hence it has a point in U, that is, $g(x) \in U$ for some $g \in G$. Then $x \in g^{-1}(U)$. We obtain that the topological space X is covered by open sets of the form $g^{-1}(U)$, where $g \in G$. Since X is compact, there exist finitely many elements $g_1, g_2, \dots, g_k \in G$ such that the sets $g_i^{-1}(U)$, $1 \le i \le k$ form a subcover. It follows that

$$\nu(g_1^{-1}(U)) + \nu(g_2^{-1}(U)) + \cdots + \nu(g_k^{-1}(U)) \ge \nu(X) = 1$$

for any Borel probability measure ν on X. If the measure ν is invariant under the action of the group G on X, then $\nu(g^{-1}(U)) = \nu(U)$ for all $g \in G$. As a consequence, $k\nu(U) \ge 1$ so that $\nu(U) \ge 1/k > 0$. $\square$

Property PE (parity exchange)**.** For any clopen set $U \subset X$ and map $f \in \mathsf{F}(G)$ there exists $g \in \mathsf{F}(G)$ such that $g(U_{\text{out}}) = U_{\text{in}}$, where $U_{\text{out}} = U \cap f^{-1}(X \setminus U)$ is the set of all points in U sent by f to the complement of U and $U_{\text{in}} = (X \setminus U) \cap f^{-1}(U)$ is the set of all points not in U sent by f to U.

Lemma 5.12. *For any ample group, Property UR implies Property PE.*

Proof. Suppose a group $G \subset \mathrm{Homeo}(X)$ is ample and has property UR. Take any clopen set $U \subset X$ and map $f \in \mathsf{F}(G)$. Note that $\mathsf{F}(G) = G$ as G is ample. Hence $f \in G$. We need to find a map $g \in \mathsf{F}(G) = G$ such that $g(U_{\text{out}}) = U_{\text{in}}$, where $U_{\text{out}} = \{x \in U \mid f(x) \notin U\}$ and $U_{\text{in}} = \{x \notin U \mid f(x) \in U\}$.

We are going to use the so-called *Kakutani-Rokhlin partition* relative to the set U and map f, which is constructed as follows. Let $U_\pm$ be the set of all points $x \in X$ such that the sequence $f(x), f^2(x), f^3(x), \dots$ visits U at least once, and so does the sequence $x, f^{-1}(x), f^{-2}(x), \dots$. For any $x \in U_\pm$ let $p(x)$ denote the least positive integer k such that $f^k(x) \in U$ and let $m(x)$ denote the least nonnegative integer k such that $f^{-k}(x) \in U$. If $p(x) > 1$ then $f(x) \in U_\pm$, $p(f(x)) = p(x) - 1$ and $m(f(x)) = m(x) + 1$. Likewise, if

$m(x) > 0$ then $f^{-1}(x) \in U_\pm$, $p(f^{-1}(x)) = p(x) + 1$ and $m(f^{-1}(x)) = m(x) - 1$. Note that $p(x) + m(x) \geq m(x) + 1 \geq 1$. For any integers k and ℓ such that $k \geq \ell \geq 1$, let $W_{k,\ell}$ denote the set of all points $x \in U_\pm$ such that $p(x) + m(x) = k$ and $m(x) + 1 = \ell$. Then the sets $W_{k,\ell}$, $k \geq \ell \geq 1$ form a partition of the set $U_\pm$. For any $k \geq 1$ the union of sets $W_{k,\ell}$, $1 \leq \ell \leq k$ is called the *tower* of height k. The set $W_{k,\ell}$ is called the ℓ-th *level* of the tower. By construction, $f(W_{k,\ell}) = W_{k,\ell+1}$ if $\ell < k$, and $f(W_{k,k}) \subset U$. That is, each level is mapped by f onto the next level of the same tower, except for the top level, which is mapped to U. Notice that the set $U \cap U_\pm$ is the union of the ground levels $W_{k,1}$, $k \geq 1$ of all towers.

Let us show that the sets $W_{k,\ell}$, $k \geq \ell \geq 1$ are clopen. If $x \in W_{k,\ell}$ then $p(x) = k - \ell + 1$ and $m(x) = \ell - 1$. Hence the points $f^{k-\ell+1}(x)$ and $f^{1-\ell}(x)$ are in U. Therefore $W_{k,\ell}$ is contained in a clopen set $\widetilde{W}_{k,\ell} = f^{\ell-k-1}(U) \cap f^{\ell-1}(U)$. Note that $\widetilde{W}_{k,\ell} \subset U_\pm$, and if $y \in \widetilde{W}_{k,\ell}$ then $p(y) \leq k - \ell + 1$ and $m(y) \leq \ell - 1$. In the case $y \notin W_{k,\ell}$, at least one of those two inequalities is strict so that $p(y) + m(y) < k$. As a consequence, the set difference $\widetilde{W}_{k,\ell} \setminus W_{k,\ell}$ is contained in the union of clopen sets of the form $\widetilde{W}_{k',\ell'}$, where $k' < k$. Observe that all such sets are disjoint from $W_{k,\ell}$. Besides, there are only finitely many of them, which implies that their union is also clopen. We conclude that the set $W_{k,\ell}$ is the difference of two clopen sets. Hence it is clopen as well.

Since the map f belongs to the group G, which has property UR, it follows that $U \subset U_\pm$ and $f(U_\pm) \subset U_\pm$. Since $f^{-1} \in G$, it follows that $f^{-1}(U_\pm) \subset U_\pm$. Hence $f(U_\pm) = U_\pm$. For any integers k and ℓ, $k \geq \ell \geq 1$, we have $W_{k,\ell} \subset U$ if $\ell = 1$ and $W_{k,\ell} \subset U_\pm \setminus U$ otherwise. It follows from the above that $f(W_{k,\ell}) \subset U$ if $\ell = k$ and $f(W_{k,\ell}) \subset U_\pm \setminus U$ otherwise. Therefore the set U_{out} is the disjoint union of sets $W_{k,1}$, $k \geq 2$ (ground levels of all towers except the tower of height 1) while the set U_{in} is the disjoint union of sets $W_{k,k}$, $k \geq 2$ (top levels of all towers except the tower of height 1). Now we define a map $g : X \to X$ as follows. For any integer $k \geq 2$, the map g coincides with f^{k-1} on $W_{k,1}$ and coincides with f^{1-k} on $W_{k,k}$. Everywhere else, g coincides with the identity function. By construction, g is invertible ($g^{-1} = g$) and $g(W_{k,1}) = W_{k,k}$ for all $k \geq 2$, which implies that $g(U_{\text{out}}) = U_{\text{in}}$.

Notice that the set U is the disjoint union of clopen sets $W_{k,1}$, $k \geq 1$. As U is compact, all but finitely many of those sets have to be empty. If $W_{k,1}$ is empty for some k, then the sets $W_{k,2}, W_{k,3}, \ldots, W_{k,k}$ are empty as well. It follows that only finitely many sets of the form $W_{k,\ell}$ are nonempty. As a consequence, the map g is piecewise an element of the group G. Since g is invertible, we conclude that $g \in \mathsf{F}(G)$. $\square$

Property E (entanglement)**.** Whenever clopen sets U_1 and U_2 intersect, the local group $\mathsf{F}_{U_1 \cup U_2}(G)$ is generated by the union of local groups $\mathsf{F}_{U_1}(G) \cup \mathsf{F}_{U_2}(G)$.

6 Generalized 2-cycles

Definition 6.1. Let U be a clopen subset of X. For any homeomorphism $f : X \to X$ such that the image $f(U)$ is disjoint from U, we define a **generalized 2-cycle** $\delta_{U;f} : X \to X$ by

$$\delta_{U;f}(x) = \begin{cases} f(x) & \text{if } x \in U, \\ f^{-1}(x) & \text{if } x \in f(U), \\ x & \text{otherwise.} \end{cases}$$

The generalized 2-cycles are a special case of the generalized permutations (introduced in Section 3). Indeed, $\delta_{U;f} = \mu[U; \mathrm{id}_X, f; (1\,2)]$. As a consequence, $\delta_{U;f} \in \mathsf{S}(G)$ whenever $f \in G$.

Recall that the symmetric group S_n is generated by all 2-cycles $(i\,j)$ in S_n.

Lemma 6.2. *For any group $G \subset \mathrm{Homeo}(X)$, a generalized 2-cycle belongs to $\mathsf{F}(G)$ if and only if it is of the form $\delta_{U;f}$, where $f \in \mathsf{F}(G)$. The generalized symmetric group $\mathsf{S}(G)$ is generated by all generalized 2-cycles in $\mathsf{F}(G)$.*

Proof. If a generalized 2-cycle $g = \delta_{U;f}$ is defined and $f \in \mathsf{F}(G)$, then g is piecewise an element of the group $\mathsf{F}(G)$. Hence it belongs to the group $\mathsf{F}(\mathsf{F}(G))$, which coincides with $\mathsf{F}(G)$ due to Lemma 2.3. Conversely, if $g = \delta_{U;f}$ is defined and $g \in \mathsf{F}(G)$, we observe that the generalized 2-cycle $\delta_{U;g}$ is also defined and $g = \delta_{U;g}$.

By Lemma 3.6, the group $\mathsf{S}(G)$ is generated by generalized permutations of the form $\mu[U; f_1, f_2, \dots, f_n; \pi]$, where $f_1, f_2, \dots, f_n \in \mathsf{F}(G)$. Those include all maps of the form $\delta_{U;f}$, $f \in \mathsf{F}(G)$. Whenever $\mu[U; f_1, f_2, \dots, f_n; \pi]$ is defined, all maps $\mu[U; f_1, f_2, \dots, f_n; \sigma]$, $\sigma \in \mathsf{S}_n$ are defined as well. Assuming $f_1, f_2, \dots, f_n \in \mathsf{F}(G)$, they all belong to the group $\mathsf{F}(\mathsf{F}(G)) = \mathsf{F}(G)$. Moreover, $\mathsf{S}_n \ni \sigma \mapsto \mu[U; f_1, f_2, \dots, f_n; \sigma]$ is a homomorphism of the group S_n to $\mathsf{F}(G)$ due to Lemma 3.4. Since the group S_n is generated by 2-cycles $(i\,j)$, it follows that the group $\mathsf{S}(G)$ is generated by generalized permutations of the form $\mu[U; f_1, f_2, \dots, f_n; (i\,j)]$. It remains to show that each $\mu[U; f_1, f_2, \dots, f_n; (i\,j)]$ is a generalized 2-cycle of prescribed form. First we observe that $\mu[U; f_1, f_2, \dots, f_n; (i\,j)] = \mu[U; f_i, f_j; (1\,2)]$. Then

$$\mu[U; f_i, f_j; (1\,2)] = \mu\big[f_i(U); \mathrm{id}_X, f_j f_i^{-1}; (1\,2)\big] = \delta_{f_i(U); f_j f_i^{-1}}$$

due to Lemma 3.3. Finally, we note that $f_j f_i^{-1} \in \mathsf{F}(G)$. $\square$

Lemma 6.3. *Suppose $U = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ is a decomposition of a clopen set U as a disjoint union of smaller clopen sets. If a generalized 2-cycle $\delta_{U;f}$ is defined then the maps $\delta_{U_i;f}$, $1 \le i \le k$ are defined as well. Moreover, they commute with one another and $\delta_{U;f} = \delta_{U_1;f}\, \delta_{U_2;f} \dots \delta_{U_k;f}$.*

Proof. For any homeomorphism $f : X \to X$, the images $f(U_1), f(U_2), \dots, f(U_k)$ are disjoint clopen sets that partition the clopen set $f(U)$. Assuming the map $\delta_{U;f}$ is defined, $f(U)$ is disjoint from U. Then the sets $U_1, U_2, \dots, U_k, f(U_1), f(U_2), \dots, f(U_k)$ are disjoint from one another. In particular, the generalized 2-cycle $\delta_{U_i;f}$ is defined for each i, $1 \le i \le k$. Since the support of $\delta_{U_i;f}$ is $U_i \cup f(U_i)$, the maps $\delta_{U_1;f}, \delta_{U_2;f}, \dots, \delta_{U_k;f}$ have pairwise disjoint supports. By Lemma 3.9, these maps commute with one another.

For any i, $1 \le i \le k$, the map $\delta_{U_i;f}$ coincides with $\delta_{U;f}$ on $U_i \cup f(U_i)$ and with the identity map everywhere else. Since sets $U_i \cup f(U_i)$, $1 \le i \le k$ form a partition of the set $U \cup f(U)$, it follows that $\delta_{U;f} = \delta_{U_1;f}\, \delta_{U_2;f} \dots \delta_{U_k;f}$. $\square$

As the first application of Lemma 6.3, we refine Lemma 6.2.

Lemma 6.4. *The generalized symmetric group $\mathsf{S}(G)$ over a group $G \subset \mathrm{Homeo}(X)$ is generated by all generalized 2-cycles of the form $\delta_{U;f}$, where $f \in G$.*

Proof. Let D be the set of all generalized 2-cycles of the form $\delta_{U;f}$, where $f \in \mathsf{F}(G)$, and D_0 be the set of all generalized 2-cycles of the form $\delta_{U;f}$, where $f \in G$. Note that $D_0 \subset D$ since $G \subset \mathsf{F}(G)$. Take any clopen set $U \subset X$ and map $f \in \mathsf{F}(G)$ such that $\delta_{U;f}$ is defined. Since f is piecewise an element of the group G, there exist clopen sets $V_1, V_2, \dots, V_k$ and maps $f_1, f_2, \dots, f_k \in G$ such that $X = V_1 \sqcup V_2 \sqcup \dots \sqcup V_k$ and f coincides with f_i on V_i, $1 \le i \le k$. Let $U_i = U \cap V_i$, $1 \le i \le k$. Then $U_1, U_2, \dots, U_k$ are clopen sets that partition U. By Lemma 6.3, $\delta_{U;f} = \delta_{U_1;f}\, \delta_{U_2;f} \dots \delta_{U_k;f}$. Since $f|_{U_i} = f_i|_{U_i}$ for $1 \le i \le k$, it follows that the map $\delta_{U_i;f_i}$ is defined and $\delta_{U_i;f} = \delta_{U_i;f_i}$ for $1 \le i \le k$. We conclude that every element of the set D can be written as a product of elements of the set D_0. Therefore the group generated by D_0 coincides with the group generated by D. By Lemma 6.2, $\langle D \rangle = \mathsf{S}(G)$. ☐

For groups with Property NSP (see Section 5), we can refine Lemma 6.2 even more.

Proposition 6.5. *Suppose S is a generating set of a group $G \subset \mathrm{Homeo}(X)$. If the group G has Property NSP, then the generalized symmetric group $\mathsf{S}(G)$ is generated by all generalized 2-cycles of the form $\delta_{U;f}$, where $f \in S$.*

Proof. Given subgroups G and H of $\mathrm{Homeo}(X)$, let $\Delta_{G,H}$ denote the set of all homeomorphisms $f \in G$ such that $\delta_{U;f} \in H$ whenever it is defined, i.e., U is a clopen set disjoint from $f(U)$. We claim that $\Delta_{G,H}$ is a subgroup of G provided that G has Property NSP.

Let us show how the proposition follows from our claim. Suppose G is a subgroup of $\mathrm{Homeo}(X)$ with Property NSP and S is a generating set for G. Let H be the subgroup of $\mathsf{S}(G)$ generated by all generalized 2-cycles of the form $\delta_{U;f}$, where $f \in S$. According to the claim, $\Delta_{G,H}$ is a subgroup of G. Since $S \subset \Delta_{G,H}$ by construction, we obtain that $\Delta_{G,H} = G$. Then it follows from Lemma 6.4 that $H = \mathsf{S}(G)$.

We proceed to the proof of the claim. First notice that $\mathrm{id}_X \in \Delta_{G,H}$. Indeed, $\delta_{U;\mathrm{id}_X}$ is defined only if U is empty, in which case $\delta_{U;\mathrm{id}_X} = \mathrm{id}_X \in H$. Next take any $f \in \Delta_{G,H}$ and assume $\delta_{U;f^{-1}}$ is defined for some clopen set $U \subset X$. Then $\delta_{f^{-1}(U),f}$ is also defined and $\delta_{U;f^{-1}} = \delta_{f^{-1}(U),f}$. As a consequence, $\delta_{U;f^{-1}} \in H$. Hence $f^{-1} \in \Delta_{G,H}$.

The hardest part is to show that for any $f, g \in \Delta_{G,H}$ we have $fg \in \Delta_{G,H}$. Take any point $x \in X$ such that $f(g(x)) \neq x$. We are going to construct a clopen neighborhood V_x of x such that $\delta_{W;fg} \in H$ for every clopen set $W \subset V_x$. First consider the case when the point $g(x)$ is different from both x and $f(g(x))$. Then there exist disjoint clopen sets U_1, U_2 and U_3 such that $x \in U_1$, $g(x) \in U_2$ and $f(g(x)) \in U_3$. Let $V_x = U_1 \cap g^{-1}(U_2) \cap (fg)^{-1}(U_3)$. The set V_x is a clopen neighborhood of x. The clopen sets V_x, $g(V_x)$ and $(fg)(V_x)$ are disjoint since $V_x \subset U_1$, $g(V_x) \subset U_2$ and $(fg)(V_x) \subset U_3$. It follows that the generalized permutation $\mu[W; \mathrm{id}_X, g, fg; \pi]$ is defined for any clopen set $W \subset V_x$ and any permutation $\pi \in \mathsf{S}_3$. Observe that $\mu[W; \mathrm{id}_X, g, fg; (1\,2)] = \delta_{W;g}$, $\mu[W; \mathrm{id}_X, g, fg; (2\,3)] = \delta_{g(W);f}$ and $\mu[W; \mathrm{id}_X, g, fg; (1\,3)] = \delta_{W;fg}$. Since $(1\,3) = (1\,2)(2\,3)(1\,2)$, it follows from Lemma 3.4 that $\delta_{W;fg} = \delta_{W;g}\, \delta_{g(W);f}\, \delta_{W;g}$. The maps $\delta_{W;g}$ and $\delta_{g(W);f}$ belong to the group H since $f, g \in \Delta_{G,H}$. Hence $\delta_{W;fg} \in H$ as well.

Now consider the case when $g(x) = x$. There still exist disjoint clopen sets U_1 and U_3 such that $x \in U_1$ and $f(g(x)) \in U_3$. Since $g \in G$ and the group G has Property NSP, the map g fixes all points in an open neighborhood U_2' of the point x. Without loss of generality, we may assume that U_2' is clopen. Then $V_x = U_1 \cap U_2' \cap (fg)^{-1}(U_3)$ is a clopen neighborhood of x. For any clopen set $W \subset V_x$, the image $(fg)(W)$ is disjoint from W since $W \subset U_1$ while $(fg)(W) \subset U_3$. Besides, the map fg coincides with f on W since $W \subset U_2'$. Hence $\delta_{W;fg}$ and $\delta_{W;f}$ both exist and $\delta_{W;fg} = \delta_{W;f}$. As $f \in \Delta_{G,H}$, we obtain that $\delta_{W;fg} \in H$.

Finally, consider the case when $g(x) = f(g(x))$. Again there exist disjoint clopen sets U_1 and U_3 such that $x \in U_1$ and $f(g(x)) \in U_3$. Since $f \in G$ and G has Property NSP, the map f fixes all points in an open neighborhood U_2'' of the point $g(x)$, which we may assume to be clopen. Then $V_x = U_1 \cap g^{-1}(U_2') \cap (fg)^{-1}(U_3)$ is a clopen neighborhood of x. For any clopen set $W \subset V_x$, the image $(fg)(W)$ is disjoint from W since $W \subset U_1$ while $(fg)(W) \subset U_3$. Besides, the map fg coincides with g on W since $g(W) \subset U_2''$. Hence $\delta_{W;fg}$ and $\delta_{W;g}$ both exist and $\delta_{W;fg} = \delta_{W;g}$. As $g \in \Delta_{G,H}$, we obtain that $\delta_{W;fg} \in H$.

Take any clopen set $U \subset X$ such that the generalized 2-cycle $\delta_{U;fg}$ is defined. Then $f(g(x)) \neq x$ for all $x \in U$. Therefore U is covered by the clopen sets V_x constructed above. Since U is compact, there are finitely many points $x_1, x_2, \dots, x_k$ such that the sets V_{x_i}, $1 \leq i \leq k$ form a subcover. Now let $W_1 = U \cap V_{x_1}$ and $W_i = (U \cap V_{x_i}) \setminus (V_{x_1} \cup \dots \cup V_{x_{i-1}})$ for $i = 2, 3, \dots, k$. Then $W_1, W_2, \dots, W_k$ are clopen sets that form a partition of U. By Lemma 6.3, $\delta_{U;fg} = \delta_{W_1;fg}\, \delta_{W_2;fg} \dots \delta_{W_k;fg}$. Since $W_i \subset V_{x_i}$ for $1 \leq i \leq k$, it follows from the above that $\delta_{W_i;fg} \in H$ for $1 \leq i \leq k$. Then $\delta_{U;fg} \in H$ as well. Thus $fg \in \Delta_{G,H}$. $\square$

7 Maximal subgroups of ample groups

In this section we study maximal subgroups of an ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$, where the topological space X is compact, metrizable, and totally disconnected. We give examples of such subgroups and obtain some partial results on their classification. Most of the time we require the group $\mathcal{G}$ to act minimally on X (Property M). Many results also require Property E (see Section 5).

Definition 7.1. A subgroup H of a group G is called **maximal** if $H \neq G$ and there is no subgroup K such that $H \subset K \subset G$ while $K \neq H$ and $K \neq G$.

The first class of subgroups to look for maximal subgroups are the stabilizers of closed sets (see Definition 4.3). Since $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}_{\mathcal{G}}(X \setminus Y)$ for any set $Y \subset X$, they are also the stabilizers of open sets.

Lemma 7.2. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y is a closed subset of X. Then $\mathrm{St}_{\mathcal{G}}(Y) \neq \mathcal{G}$ unless Y is the empty set or $Y = X$.*

Proof. Clearly, the entire group $\mathcal{G}$ is the stabilizer of X and of the empty set. If Y is not empty, we can find a point $x \in Y$. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in X (due to Lemma 5.9). If $Y \neq X$ then $X \setminus Y$ is a nonempty open set. Hence $\mathrm{Orb}_{\mathcal{G}}(x)$ has a point in $X \setminus Y$. This means that $g(x) \notin Y$ for some $g \in \mathcal{G}$. Then $g \notin \mathrm{St}_{\mathcal{G}}(Y)$, which implies that $\mathrm{St}_{\mathcal{G}}(Y) \neq \mathcal{G}$. $\square$

Lemma 7.3. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group and $x, y \in X$ be two points that lie in the same orbit of $\mathcal{G}$. Suppose Y_1 is a subset of X such that $x \in Y_1$ while $y \notin Y_1$. Suppose Y_2 is a subset of X such that x and y are both interior points of Y_2. Then $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$.*

Proof. Clearly, $x \neq y$. Since the topological space X is totally disconnected, there exists a clopen neighborhood U of the point x that does not contain the point y. Since X is totally disconnected and compact, it follows that any open neighborhood of any point contains a clopen neighborhood of the same point. As a consequence, the set Y_2 contains a clopen neighborhood U_1 of x and a clopen neighborhood U_2 of y. Let $V_1 = U_1 \cap U$ and $V_2 = U_2 \cap (X \setminus U)$. Then V_1 and V_2 are disjoint clopen neighborhoods of respectively x and y. Since the points x and y lie in the same orbit of the group $\mathcal{G}$, we have $y = g(x)$ for some $g \in \mathcal{G}$. Let $W = V_1 \cap g^{-1}(V_2)$. Since g is a homeomorphism, it follows that W is a clopen neighborhood of x. The image $g(W)$ is disjoint from W as $W \subset V_1$ while $g(W) \subset V_2$. Therefore the generalized 2-cycle $\delta_{W;g}$ is defined. Since $g \in \mathcal{G}$, the map $\delta_{W;g}$ is piecewise an element of the group $\mathcal{G}$. Hence $\delta_{W;g}$ belongs to the group $\mathsf{F}(\mathcal{G})$, which coincides with $\mathcal{G}$ as $\mathcal{G}$ is ample. The homeomorphism $\delta_{W;g}$ maps the sets W and $g(W)$ onto each other while fixing all points not in $W \cup g(W)$. Since $W \cup g(W) \subset V_1 \cup V_2 \subset U_1 \cup U_2 \subset Y_2$, we obtain that $\delta_{W;g}(Y_2) = Y_2$. Hence $\delta_{W;g} \in \mathrm{St}_{\mathcal{G}}(Y_2)$. At the same time, $\delta_{W;g}$ maps the point x, which belongs to Y_1, to the point $g(x) = y$, which is not in Y_1. Hence $\delta_{W;g} \notin \mathrm{St}_{\mathcal{G}}(Y_1)$. We conclude that $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$. □

Lemma 7.4. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y_1 and Y_2 are distinct closed subsets of X. Then $\mathrm{St}_{\mathcal{G}}(Y_1) = \mathrm{St}_{\mathcal{G}}(Y_2)$ if and only if $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. Equivalent conditions are that $Y_1 = \overline{X \setminus Y_2}$ and $Y_2 = \overline{X \setminus Y_1}$.*

Proof. We have $f(X \setminus Y) = X \setminus f(Y)$ for any invertible map $f : X \to X$ and any set $Y \subset X$. Therefore $f(X \setminus Y) = X \setminus Y$ if and only if $f(Y) = Y$. As a consequence, $\mathrm{St}_{\mathcal{G}}(X \setminus Y) = \mathrm{St}_{\mathcal{G}}(Y)$. If f is a homeomorphism, then also $f(\overline{Y}) = \overline{f(Y)}$. Hence $f(\overline{Y}) = \overline{Y}$ whenever $f(Y) = Y$. As a consequence, $\mathrm{St}_{\mathcal{G}}(Y) \subset \mathrm{St}_{\mathcal{G}}(\overline{Y})$.

Assume that sets $Y_1, Y_2 \subset X$ satisfy conditions $Y_1 = \overline{X \setminus Y_2}$ and $Y_2 = \overline{X \setminus Y_1}$. By the above, $\mathrm{St}_{\mathcal{G}}(Y_2) = \mathrm{St}_{\mathcal{G}}(X \setminus Y_2) \subset \mathrm{St}_{\mathcal{G}}(\overline{X \setminus Y_2}) = \mathrm{St}_{\mathcal{G}}(Y_1)$. Likewise, $\mathrm{St}_{\mathcal{G}}(Y_1) \subset \mathrm{St}_{\mathcal{G}}(Y_2)$. Hence $\mathrm{St}_{\mathcal{G}}(Y_1) = \mathrm{St}_{\mathcal{G}}(Y_2)$. Further, either of the two assumed conditions implies that $Y_1 \cup Y_2 = X$. The first condition also implies that no interior point of Y_2 belongs to Y_1. Likewise, the second of the two implies that no interior point of Y_1 belongs to Y_2. Therefore any common point of Y_1 and Y_2 is their common boundary point: $Y_1 \cap Y_2 \subset \partial Y_1 \cap \partial Y_2$. Since the sets Y_1 and Y_2 are clearly closed, they contain their own boundaries so that $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. Conversely, assume $Y_1, Y_2 \subset X$ are closed sets such that $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. Then X is a disjoint union of three sets $Y_1 \setminus Y_2$, $Y_2 \setminus Y_1$ and $Y_1 \cap Y_2$. Moreover, the sets $Y_1 \setminus Y_2 = X \setminus Y_2$ and $Y_2 \setminus Y_1 = X \setminus Y_1$ are open. It follows that $Y_1 \setminus Y_2$ and $Y_2 \setminus Y_1$ are disjoint from both ∂Y_1 and ∂Y_2. Hence $\partial Y_1 \cup \partial Y_2 \subset Y_1 \cap Y_2$, which implies that $\partial Y_1 = \partial Y_2 = Y_1 \cap Y_2$. Since $\partial Y_1 = \partial(X \setminus Y_1) = \partial(Y_2 \setminus Y_1)$, we obtain that $\overline{X \setminus Y_1} = \overline{Y_2 \setminus Y_1} = (Y_2 \setminus Y_1) \cup \partial(Y_2 \setminus Y_1) = (Y_2 \setminus Y_1) \cup (Y_1 \cap Y_2) = Y_2$. Likewise, $\overline{X \setminus Y_2} = Y_1$.

It remains to show that $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$ for any distinct closed sets $Y_1, Y_2 \subset X$ that do not satisfy at least one of the conditions $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. First we consider the case when $Y_1 \cup Y_2 \neq X$. Then $U = X \setminus (Y_1 \cup Y_2)$ is a nonempty open set. Since $Y_1 \neq Y_2$, there is a point $x \in X$ that belongs to one of these sets but not to the other. We may assume without loss of generality that $x \in Y_1$ and $x \notin Y_2$. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in X (due to Lemma 5.9). In particular, it has a point y in U. Clearly, $y \notin Y_1$. Besides, neither x nor y belongs to Y_2. Therefore x and y are both interior points of the open set $X \setminus Y_2$. By Lemma 7.3, $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(X \setminus Y_2)$. Since $\mathrm{St}_{\mathcal{G}}(X \setminus Y_2) = \mathrm{St}_{\mathcal{G}}(Y_2)$, we also have that $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$.

Now consider the case when $Y_1 \cup Y_2 = X$ but there exists a point $x \in Y_1 \cap Y_2$ that is not a boundary point for at least one of the sets Y_1 and Y_2. We may assume without loss of generality that x is an interior point of Y_2. Furthermore, we may assume that $Y_1 \neq X$ as otherwise $\mathrm{St}_{\mathcal{G}}(Y_2) \neq \mathcal{G} = \mathrm{St}_{\mathcal{G}}(Y_1)$ due to Lemma 7.2. Then $X \setminus Y_1$ is a nonempty open set. Just as above, minimality of the action of the group $\mathcal{G}$ on X implies that the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ has a point y in $X \setminus Y_1$. Note that $X \setminus Y_1 \subset Y_2$ since $Y_1 \cup Y_2 = X$. Therefore every point of $X \setminus Y_1$ (including y) is an interior point of Y_2. We obtain that $x \in Y_1$, $y \notin Y_1$, and both x and y are interior points of Y_2. By Lemma 7.3, $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$. □

Lemma 7.5. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y_1 and Y_2 are nonempty closed subsets of X such that $Y_1 \subset Y_2$ and $Y_1 \neq Y_2$. Then $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$.*

Proof. If $Y_2 = X$ then $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathcal{G} = \mathrm{St}_{\mathcal{G}}(Y_2)$ due to Lemma 7.2. Otherwise $Y_1 \cup Y_2 = Y_2 \neq X$. Then $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$ due to Lemma 7.4. □

Now we are ready to establish our first result on the classification of maximal subgroups of ample groups.

Theorem 7.6. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a maximal subgroup of $\mathcal{G}$ that does not act minimally on X. Then $H = \mathrm{St}_{\mathcal{G}}(Y)$ for some closed set $Y \subset X$ different from the empty set and X. Furthermore, if the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set $Y \subset X$ is indeed a maximal subgroup of $\mathcal{G}$, then the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal.*

Proof. Since the group H does not act minimally on X, it admits a closed invariant set Y different from the empty set and X. We have $h(Y) = Y$ for any $h \in H$. Hence $H \subset \mathrm{St}_{\mathcal{G}}(Y)$. Lemma 7.2 implies that $\mathrm{St}_{\mathcal{G}}(Y) \neq \mathcal{G}$. As H is a maximal subgroup of $\mathcal{G}$, it follows that $H = \mathrm{St}_{\mathcal{G}}(Y)$.

Now assume that a group $H = \mathrm{St}_{\mathcal{G}}(Y)$, where Y is a closed subset of X, is indeed a maximal subgroup of $\mathcal{G}$. To prove that the induced action of H on Y is minimal, we need to show that any closed set $Y_0 \subset Y$ invariant under it coincides with the empty set or Y. We have $h(Y_0) = Y_0$ for any $h \in H$. Hence $H \subset \mathrm{St}_{\mathcal{G}}(Y_0)$. Note that $Y \neq X$ since $H \neq \mathcal{G}$. Then $Y_0 \neq X$ as well. Assume that $Y_0 \neq \emptyset$. Just as above, Lemma 7.2 implies that $\mathrm{St}_{\mathcal{G}}(Y_0) \neq \mathcal{G}$ and then maximality of H implies that $H = \mathrm{St}_{\mathcal{G}}(Y_0)$. Since $\mathrm{St}_{\mathcal{G}}(Y_0) = \mathrm{St}_{\mathcal{G}}(Y)$ and $Y_0 \subset Y$, it follows from Lemma 7.5 that $Y_0 = Y$. □

The assumption that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set Y acts minimally when restricted to Y has implications for the set Y and for the group $\mathrm{St}_{\mathcal{G}}(Y)$.

Lemma 7.7. *Let G be a subgroup of $\mathrm{Homeo}(X)$. Suppose Y is a closed subset of X such that the stabilizer $\mathrm{St}_G(Y)$ acts minimally when restricted to Y. Then the set Y is either clopen or nowhere dense in X.*

Proof. We have $h(Y) = Y$ for any $h \in \mathrm{St}_G(Y)$. Since h is a homeomorphism of the ambient space X, it maps interior points of the set Y to its interior points and boundary points of Y to its boundary points. Therefore the boundary ∂Y, which is a closed subset of Y, is invariant under the induced action of the group $\mathrm{St}_G(Y)$ on Y. As the latter action is minimal, the boundary ∂Y has to coincide with the empty set or Y. If $\partial Y = \emptyset$ then the set Y is clopen. If $\partial Y = Y$ then Y is nowhere dense in X. $\square$

Lemma 7.8. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose U is a nonempty clopen subset of X. Then any orbit of the induced action of the stabilizer $\mathrm{St}_{\mathcal{G}}(U)$ on U is the intersection of an orbit of $\mathcal{G}$ with U. As a consequence, $\mathrm{St}_{\mathcal{G}}(U)$ acts minimally on U whenever $\mathcal{G}$ acts minimally on X.*

Proof. The induced action of the group $H = \mathrm{St}_{\mathcal{G}}(U)$ on U is the restriction of its action on X. Therefore every orbit of the induced action is of the form $\mathrm{Orb}_H(x)$, where $x \in U$. We need to show that $\mathrm{Orb}_H(x) = \mathrm{Orb}_{\mathcal{G}}(x) \cap U$. The inclusion $\mathrm{Orb}_H(x) \subset \mathrm{Orb}_{\mathcal{G}}(x) \cap U$ is obvious. Conversely, take any point $y \in \mathrm{Orb}_{\mathcal{G}}(x) \cap U$ different from x. We have $y = g(x)$ for some $g \in \mathcal{G}$. Since the topological space X is totally disconnected, the point x has a clopen neighborhood U_0 that does not contain y. Then $U \cap U_0$ and $U \setminus U_0$ are clopen neighborhoods of respectively x and y. Since g is a continuous map, the set $W = U \cap U_0 \cap g^{-1}(U \setminus U_0)$ is a clopen neighborhood of x. The image $g(W)$ is disjoint from W as $W \subset U \cap U_0$ while $g(W) \subset U \setminus U_0$. Therefore the generalized 2-cycle $\delta_{W;g}$ is defined. Since $g \in \mathcal{G}$, the map $\delta_{W;g}$ belongs to the group $\mathsf{F}(\mathcal{G}) = \mathcal{G}$. The homeomorphism $\delta_{W;g}$ maps the sets W and $g(W)$ onto each other while fixing all points not in $W \cup g(W)$. Since $W \cup g(W) \subset U$, we obtain that $\delta_{W;g}(U) = U$. Hence $\delta_{W;g} \in H$. By construction, $\delta_{W;g}(x) = g(x) = y$. Thus $y \in \mathrm{Orb}_H(x)$.

Assuming the group $\mathcal{G}$ acts minimally on X, every orbit of $\mathcal{G}$ is dense in X (due to Lemma 5.9). Since U is a clopen set, it follows that the intersection of any orbit of $\mathcal{G}$ with U is dense in U. By the above this means that every orbit of the induced action of the group H on U is dense in U. Hence the latter action is minimal (again due to Lemma 5.9). $\square$

Lemma 7.9. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y is a closed subset of X such that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y. Then for any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(Y)$ the group $\langle \mathrm{St}_{\mathcal{G}}(Y) \cup \{g\}\rangle$ generated by $\mathrm{St}_{\mathcal{G}}(Y)$ and g has the same orbits as $\mathcal{G}$.*

Proof. Let $H = \mathrm{St}_{\mathcal{G}}(Y)$. For any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(Y)$, let $H_g = \langle \mathrm{St}_{\mathcal{G}}(Y) \cup \{g\}\rangle$. First let us show that no orbit of the group H_g is contained in Y. Since g does not map the set Y onto itself, there exists a point $x_0 \in Y$ such that $g(x_0) \notin Y$ or $g^{-1}(x_0) \notin Y$. In either case, we have an element $g_0 \in H_g$ such that $g_0(x_0) \notin Y$. Note that $X \setminus Y$ is an open neighborhood of the point $g_0(x_0)$. It follows that the set $U = g_0^{-1}(X \setminus Y)$ is an open neighborhood of x_0. Take any $x \in Y$. Since the group H acts minimally when restricted to Y, the orbit $\mathrm{Orb}_H(x)$ is dense in Y. In particular, it has a point in $U \cap Y$. That is, $h(x) \in U$ for some $h \in H$. Then $g_0 h \in H_g$ and $g_0 h(x) \in g_0(U) = X \setminus Y$. Hence the orbit $\mathrm{Orb}_{H_g}(x)$ has a point outside Y.

Since H_g is a subgroup of $\mathcal{G}$, for any $x \in X$ the orbit $\mathrm{Orb}_{H_g}(x)$ is contained in $\mathrm{Orb}_{\mathcal{G}}(x)$. We need to show that, conversely, $\mathrm{Orb}_{\mathcal{G}}(x) \subset \mathrm{Orb}_{H_g}(x)$. Take any point $y \in \mathrm{Orb}_{\mathcal{G}}(x)$. By the above the orbits of the points x and y under the action of the group H_g are not contained in

Y. Hence there exist $h_1, h_2 \in H_g$ such that the points $x_1 = h_1(x)$ and $y_1 = h_2(y)$ are outside the set Y. If $x_1 = y_1$ then $y = h_2^{-1}h_1(x)$ so that $y \in \mathrm{Orb}_{H_g}(x)$. Now assume $x_1 \neq y_1$. Both x_1 and y_1 do belong to $\mathrm{Orb}_{\mathcal{G}}(x)$. Therefore $y_1 = g_1(x_1)$ for some $g_1 \in \mathcal{G}$. Note that $X \setminus Y$ is an open neighborhood of both x_1 and y_1. Since the topological space X is totally disconnected and compact, it follows that the points x_1 and y_1 have some clopen neighborhoods U_1 and U_2, respectively, contained in $X \setminus Y$. Besides, x_1 has a clopen neighborhood U_0 that does not include y_1. Then $U_0 \cap U_1$ is a clopen neighborhood of x_1 and $U_2 \setminus U_0$ is a clopen neighborhood of y_1. Consequently, the set $W = U_0 \cap U_1 \cap g_1^{-1}(U_2 \setminus U_0)$ is yet another clopen neighborhood of x_1. The image $g_1(W)$ is disjoint from W since $W \subset U_0 \cap U_1$ while $g_1(W) \subset U_2 \setminus U_0$. Therefore a generalized 2-cycle $f = \delta_{W;g_1}$ is defined. Since $g_1 \in \mathcal{G}$, the map f belongs to the group $\mathsf{F}(\mathcal{G}) = \mathcal{G}$. The map f fixes all points not in $W \cup g_1(W) \subset U_1 \cup U_2$. In particular, it fixes every point of the set Y so that $f \in \mathrm{St}^{\star}_{\mathcal{G}}(Y) \subset \mathrm{St}_{\mathcal{G}}(Y) \subset H_g$. By construction, $f(x_1) = g_1(x_1) = y_1$. Then $y = h_2^{-1} f h_1(x)$. Since $h_2^{-1} f h_1 \in H_g$, we conclude that $y \in \mathrm{Orb}_{H_g}(x)$. $\square$

The simplest example of a closed subset of X is a finite set. To treat the stabilizers of finite sets, we do not even need to assume that the ample group acts minimally on X.

Lemma 7.10. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose $Y \subset X$ is a finite set contained in a single orbit of $\mathcal{G}$ and $Z \subset X$ is a finite set disjoint from Y. Then for any permutation $\pi : Y \to Y$ there exists a map $f \in \mathcal{G}$ such that $f(x) = \pi(x)$ for all $x \in Y$ and $f(x) = x$ for all $x \in Z$.*

Proof. First we consider a particular case when the permutation π is a transposition. That is, $\pi = (x\,y)$, where x and y are distinct elements of Y. Since the topological space X is totally disconnected, for any point $z \in Y \cup Z$ different from x there exists a clopen neighborhood U_z of x that does not contain z. Let U denote the intersection of the sets U_z over all points $z \in Y \cup Z$ different from x. The set U is clopen since Y and Z are finite sets. By construction, x is the only element of $Y \cup Z$ that belongs to U. Similarly, there exists a clopen neighborhood V of the point y that contains no other element of $Y \cup Z$. Notice that the points x and y are in the same orbit of the group $\mathcal{G}$. Hence $y = g(x)$ for some $g \in \mathcal{G}$. Let $W = U \cap g^{-1}(V \setminus U)$. Observe that $V \setminus U$ is a clopen set containing y. Since g is a continuous map, it follows that W is a clopen neighborhood of x. Furthermore, the set W is disjoint from $g(W)$ as $W \subset U$ while $g(W) \subset V \setminus U$. Therefore the generalized 2-cycle $f = \delta_{W;g}$ is defined. Since $g \in \mathcal{G}$, it follows that $f \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. Clearly, $f(x) = y$ and $f(y) = x$. All the other elements of $Y \cup Z$ are fixed by f since they do not belong to the set $W \cup g(W) \subset U \cup V$. Hence $f(z) = \pi(z)$ for all $z \in Y$ and $f(z) = z$ for all $z \in Z$.

We proceed to the general case. If π is the identity map, we can take $f = \mathrm{id}_X$, which belongs to $\mathcal{G}$. Otherwise the permutation π can be decomposed as a product, $\pi = \pi_1\pi_2 \dots \pi_k$, where each π_i is a transposition exchanging two elements of Y. By the above for any i, $1 \leq i \leq k$ there exists a map $f_i \in \mathcal{G}$ that coincides with π_i on the set Y and fixes all points of the set Z. Then $f = f_1 f_2 \dots f_k$ is an element of $\mathcal{G}$ that coincides with the permutation $\pi_1\pi_2 \dots \pi_k = \pi$ on Y and fixes all points of Z. $\square$

Lemma 7.11. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose $Y \subset X$ is a finite set and $\pi : Y \to Y$ is a permutation that maps any point of Y to a point in the same orbit of $\mathcal{G}$. Then there exists a map $f \in \mathcal{G}$ such that $f(x) = \pi(x)$ for all $x \in Y$.*

Proof. The orbits of $\mathcal{G}$ form a partition of the set X, which induces a partition of its subset Y. Namely, $Y = Y_1 \sqcup Y_2 \sqcup \dots \sqcup Y_k$, where each of the sets Y_i, $1 \le i \le k$ is contained in a single orbit of $\mathcal{G}$ and no two sets are contained in the same orbit. Then each of the sets Y_i, $1 \le i \le k$ is invariant under the permutation π. Hence there exists a permutation $\pi_i : Y \to Y$ that coincides with π on Y_i while fixing all points in $Y \setminus Y_i$. Since Y_i is contained in a single orbit of $\mathcal{G}$, Lemma 7.10 implies that there exists a map $f_i \in \mathcal{G}$ that coincides with π_i on Y. Then $f = f_1 f_2 \dots f_k$ is an element of $\mathcal{G}$ that coincides with the permutation $\pi_1\pi_2\dots\pi_k$ on the set Y. It is easy to observe that $\pi_1\pi_2\dots\pi_k = \pi$. $\square$

The next theorem is our main result on the stabilizers of finite sets.

Theorem 7.12. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose Y is a finite nonempty subset of X. Then the following statements hold true.*

(i) If Y is a proper subset of a single orbit of $\mathcal{G}$ and, moreover, Y does not contain exactly half of points in the orbit, then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$.

(ii) If Y is a subset of an orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ that contains exactly half of its points, then $\mathrm{St}_{\mathcal{G}}(Y)$ is a subgroup of index 2 in the group $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. The latter is a maximal subgroup of $\mathcal{G}$ unless Y consists of a single point, in which case $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x)\setminus Y) = \mathcal{G}$.

(iii) If there are at least two orbits of $\mathcal{G}$ that intersect Y but are not contained in Y, then $\mathrm{St}_{\mathcal{G}}(Y)$ is a proper subgroup of $\mathcal{G}$ that is not maximal.

(iv) If Y is a union of orbits of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$.

(v) If $Y = Y_1 \sqcup Y_2$, where Y_2 is a union of orbits of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}_{\mathcal{G}}(Y_1)$.

Proof. Since $g(h(x)) = (gh)(x)$ for all maps $g, h \in \mathcal{G}$ and points $x \in X$, it follows that $g(\mathrm{Orb}_{\mathcal{G}}(x)) \subset \mathrm{Orb}_{\mathcal{G}}(x)$. For any $g \in \mathcal{G}$, the inverse map g^{-1} is also in $\mathcal{G}$. Hence we also have $g^{-1}(\mathrm{Orb}_{\mathcal{G}}(x)) \subset \mathrm{Orb}_{\mathcal{G}}(x)$, which implies that, in fact, $g(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$. It further follows that $g(Y) = Y$ for any set $Y \subset X$ that is a union of orbits of the group $\mathcal{G}$. Consequently, $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$ for any such set Y, finite or not. Further, if a set $Y \subset X$ is represented as a disjoint union of two sets, $Y = Y_1 \sqcup Y_2$, then $g(Y) = g(Y_1) \sqcup g(Y_2)$ for any $g \in \mathcal{G}$. Assuming Y_2 is a union of orbits of $\mathcal{G}$, we have $g(Y_2) = Y_2$. Hence $g(Y) = Y$ if and only if $g(Y_1) = Y_1$. As a result, $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}_{\mathcal{G}}(Y_1)$.

We have proved statements (iv) and (v). Next we prove statement (iii). Suppose $\mathrm{Orb}_{\mathcal{G}}(x)$ and $\mathrm{Orb}_{\mathcal{G}}(y)$ are two distinct orbits of $\mathcal{G}$ that intersect a finite set $Y \subset X$ but are not contained in Y. Then there exist points $x_1, x_2 \in \mathrm{Orb}_{\mathcal{G}}(x)$ and $y_1, y_2 \in \mathrm{Orb}_{\mathcal{G}}(y)$ such that x_1 and y_1 belong to Y while x_2 and y_2 do not. By Lemma 7.10, the group $\mathcal{G}$ contains a map f that interchanges y_1 and y_2 while fixing all points of Y different from y_1. Note that $f(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(x)$ while $f(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) \ne Y \cap \mathrm{Orb}_{\mathcal{G}}(y)$. Likewise, there exists $h \in \mathcal{G}$ that interchanges x_1 and x_2 while fixing all points of Y different from x_1. Then $h(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(y)$ while $h(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) \ne Y \cap \mathrm{Orb}_{\mathcal{G}}(x)$. It follows that neither of the stabilizers $\mathrm{St}_{\mathcal{G}}(Y \cap \mathrm{Orb}_{\mathcal{G}}(x))$ and $\mathrm{St}_{\mathcal{G}}(Y \cap \mathrm{Orb}_{\mathcal{G}}(y))$ contains the other. Therefore both stabilizers are proper subgroups of $\mathcal{G}$ and their intersection is a proper subgroup of either of them. It remains to observe that the said intersection contains $\mathrm{St}_{\mathcal{G}}(Y)$. Indeed, any $g \in \mathcal{G}$ maps each orbit of $\mathcal{G}$ onto itself. Hence $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) = g(Y) \cap \mathrm{Orb}_{\mathcal{G}}(x)$ and $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) = g(Y) \cap \mathrm{Orb}_{\mathcal{G}}(y)$. Consequently, the equality $g(Y) = Y$ implies that $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(x)$ and $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(y)$.

We proceed to the main part of the theorem, namely, statements (i) and (ii). Suppose that a finite nonempty set Y is a proper subset of a single orbit $\mathrm{Orb}_{\mathcal{G}}(x)$. Since $g(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$ for any $g \in \mathcal{G}$, we have $g(Y) = Y$ if and only if $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. It follows that $\mathrm{St}_{\mathcal{G}}(Y)$ coincides with another set stabilizer $\mathrm{St}_{\mathcal{G}}(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ as well as with the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. The collective stabilizer $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ acts on the set $\{Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y\}$ and $\mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is the kernel of this action. If the set Y does not contain exactly half of points in $\mathrm{Orb}_{\mathcal{G}}(x)$, it cannot be mapped onto $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ by any invertible transformation of X. Then the action is trivial so that $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{St}_{\mathcal{G}}(Y)$. In the case when Y contains exactly half of points in $\mathrm{Orb}_{\mathcal{G}}(x)$, the orbit is finite and the set $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ contains the same number of points as Y. Then there exists a permutation π on $\mathrm{Orb}_{\mathcal{G}}(x)$ such that $\pi(Y) = \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ and $\pi(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = Y$. Lemma 7.10 implies that there is also a homeomorphism $f \in \mathcal{G}$ such that $f(Y) = \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ and $f(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = Y$. Hence the action of the group $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ on the set $\{Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y\}$ is nontrivial, which implies that $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$.

Both Y and $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ are nonempty sets. If each of them consists of a single point, then $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{St}_{\mathcal{G}}(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathcal{G}$. Otherwise we can choose two distinct points x_1 and x_2 in one of the two sets and another point x_3 in the other set. By Lemma 7.10, the ample group $\mathcal{G}$ contains a map f such that $f(x_1) = x_1$, $f(x_2) = x_3$ and $f(x_3) = x_2$. Then one of the points $f(x_1)$ and $f(x_2)$ is in Y while the other one is in $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. It follows that $f \notin \mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. Hence $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \neq \mathcal{G}$.

It remains to prove that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is a maximal subgroup of $\mathcal{G}$ provided that at least one of the sets Y and $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ consists of more than one point. We have already shown that under this condition, $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is a proper subgroup of $\mathcal{G}$. For convenience, let us assume that the number of points in Y is less than or equal to the number of points in $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ (note that the latter can be infinite). There is no loss of generality here as otherwise we could replace Y by $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Given a map $g \in \mathcal{G}$ that does not belong to $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$, let H_g denote the subgroup of $\mathcal{G}$ generated by $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ and g. We need to show that $H_g = \mathcal{G}$. This requires some preparation.

Since $g(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$, the condition $g \notin \mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is equivalent to a pair of conditions $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \neq Y$ and $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \neq \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Note that the image $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ cannot be a proper subset of Y as any proper subset of Y has strictly fewer points than $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Besides, $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ cannot be a proper subset of $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ as that would imply that Y is a proper subset of $g(Y)$, which would contradict the fact that the set Y is finite. We conclude that $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ intersects both Y and $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Hence we can choose points $x_+, x_- \in \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ such that $g(x_+) \in Y$ while $g(x_-) \in \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. By Lemma 7.10, there exists a map $h \in \mathcal{G}$ such that the restriction of h to the set $Z = Y \cup g^{-1}(Y) \cup \{x_+, x_-\}$ coincides with the transposition $(x_+\ x_-)$. Then the restriction of the map $\tilde{g} = ghg^{-1}$ to the set $g(Z) = Y \cup g(Y) \cup \{g(x_+), g(x_-)\}$ coincides with the transposition $\big(g(x_+)\, g(x_-)\big)$. Since the map h fixes all points in Y, we have $h \in \mathrm{St}_{\mathcal{G}}(Y) \subset \mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. It follows that $\tilde{g} \in H_g$. Let $y_+ = g(x_+)$ and

$y_- = g(x_-)$. By construction, $y_+ \in Y$, $y_- \in \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$, $\tilde{g}(y_+) = y_-$, $\tilde{g}(y_-) = y_+$, and $\tilde{g}(y) = y$ for all points $y \in Y$ different from y_+.

Next we prove that for any finite set $S \subset \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ containing the point y_-, there exists a map $h_S \in H_g$ such that the restriction of h_S to $Y \cup S$ coincides with the transposition $(y_+ \, y_-)$. The proof is by induction on the number n of points in S. If $n = 1$ then $S = \{y_-\}$ and we can let $h_S = \tilde{g}$. In the case $n \geq 2$, we assume the claim holds for all sets of $n-1$ points. Take any point $z_0 \in S$ different from y_-. Then the set $S_0 = S \setminus \{z_0\}$ consists of $n-1$ points including y_-. By the inductive assumption, there exists a map $h_{S_0} \in H_g$ such that the restriction of h_{S_0} to the set $Y \cup S_0$ coincides with the transposition $(y_+ \, y_-)$. If the point $\tilde{z}_0 = h_{S_0}(z_0)$ coincides with z_0, then we can clearly let $h_S = h_{S_0}$. Otherwise Lemma 7.10 implies existence of a map $\phi \in \mathcal{G}$ such that its restriction to the set $Y \cup S \cup \{\tilde{z}_0\}$ coincides with the transposition $(z_0 \, \tilde{z}_0)$. Since ϕ fixes all points in Y, it belongs to the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$, which is contained in H_g. Then the map ϕh_{S_0} is in H_g as well. By construction, $\phi h_{S_0}(z_0) = z_0$. Note that $\tilde{z}_0 \notin Y \cup S_0$ as $Y \cup S_0 = h_{S_0}(Y \cup S_0)$. Hence ϕh_{S_0} coincides with h_{S_0} on $Y \cup S_0$. Therefore we can let $h_S = \phi h_{S_0}$ and complete the inductive step.

Next we prove that for any pair of sets $S' \subset Y$ and $S'' \subset \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ of the same cardinality, there exists a map $h_{S',S''} \in H_g$ such that $h_{S',S''}(S') = S''$, $h^2_{S',S''}(y) = y$ for all $y \in S'$, and $h_{S',S''}(y) = y$ for all $y \in Y \setminus S'$. The proof is by induction on the number n of points in S'. If $n = 0$ then both sets are empty and we can let $h_{S',S''} = \mathrm{id}_X$. In the case $n \geq 1$, we assume the claim holds whenever the sets consist of $n-1$ points. Take any points $y_0 \in S'$ and $z_0 \in S''$. Then the sets $S'_0 = S' \setminus \{y_0\}$ and $S''_0 = S'' \setminus \{z_0\}$ contain $n-1$ points each. By the inductive assumption, there exists a map $h_{S'_0,S''_0} \in H_g$ such that $h_{S'_0,S''_0}(S'_0) = S''_0$, $h^2_{S'_0,S''_0}(y) = y$ for all $y \in S'_0$, and $h_{S'_0,S''_0}(y) = y$ for all $y \in Y \setminus S'_0$. Note that $h_{S'_0,S''_0}(S''_0) = S'_0$ and hence $h_{S'_0,S''_0}(Y \cup S''_0) = Y \cup S''_0$. As a consequence, the point $\tilde{z}_0 = h_{S'_0,S''_0}(z_0)$ is in $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ but not in S''_0. Consider a set $Z = S''_0 \cup \{y_-\} \cup \{z_0\} \cup \{\tilde{z}_0\}$, which is a finite subset of $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. By the above there exists a map $h_Z \in H_g$ such that its restriction to $Y \cup Z$ coincides with the transposition $(y_+ \, y_-)$. Further, by Lemma 7.11 (or Lemma 7.10), for any permutation on the set $Y \cup Z$ there exists an element of the group $\mathcal{G}$ that coincides with this permutation on $Y \cup Z$. In particular, there exists a map $\phi \in \mathcal{G}$ such that $\phi(z_0) = \tilde{z}_0$, $\phi(\tilde{z}_0) = z_0$, and $\phi(y) = y$ for all other points $y \in Y \cup Z$. The restriction of ϕ to $Y \cup Z$ coincides with the transposition $(z_0 \, \tilde{z}_0)$ if $z_0 \neq \tilde{z}_0$, and with the identity map if $z_0 = \tilde{z}_0$. Furthermore, there exists a map $\psi \in \mathcal{G}$ such that $\psi(y_+) = y_0$, $\psi(y_0) = y_+$, $\psi(y_-) = z_0$, $\psi(z_0) = y_-$, and $\psi(y) = y$ for all other points $y \in Y \cup Z$. The restriction of ψ to the set $Y \cup Z$ coincides with the permutation $(y_+ \, y_0)(y_- \, z_0)$ if $y_+ \neq y_0$ and $y_- \neq z_0$, with the transposition $(y_+ \, y_0)$ if $y_+ \neq y_0$ and $y_- = z_0$, with the transposition $(y_- \, z_0)$ if $y_+ = y_0$ and $y_- \neq z_0$, and with the identity map if $y_+ = y_0$ and $y_- = z_0$. It follows that the restriction of the map $\psi h_Z \psi^{-1}$ to $Y \cup Z$ coincides with the transposition $\big(\psi(y_-) \, \psi(y_+)\big) = (y_0 \, z_0)$. Finally, let $h_{S',S''} = \psi h_Z \psi^{-1} \phi h_{S'_0,S''_0}$. By construction, $\phi(Y) = \psi(Y) = Y$ so that $\phi, \psi \in \mathrm{St}_{\mathcal{G}}(Y) \subset H_g$. Then $h_{S',S''} \in H_g$ as well. The map $h_{S',S''}$ coincides with $h_{S'_0,S''_0}$ on the set $(Y \setminus \{y_0\}) \cup S''_0$. Besides, $h_{S',S''}(y_0) = \psi h_Z \psi^{-1} \phi(y_0) = \psi h_Z \psi^{-1}(y_0) = z_0$ and $h_{S',S''}(z_0) = \psi h_Z \psi^{-1} \phi(\tilde{z}_0) = \psi h_Z \psi^{-1}(z_0) = y_0$. The inductive step is complete.

Now we are ready to show that $H_g = \mathcal{G}$. Take any map $f \in \mathcal{G}$. We define three sets $S_{++} = Y \cap f^{-1}(Y)$, $S_{+-} = Y \cap f^{-1}(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ and $S_{-+} = (\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \cap f^{-1}(Y)$. Since

$f(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$, it follows that $Y = S_{++} \sqcup S_{+-} = f(S_{++}) \sqcup f(S_{-+})$. Since f is a one-to-one map, the set $f(S_{++})$ is of the same cardinality as S_{++} while the set $f(S_{-+})$ is of the same cardinality as S_{-+}. Since the set Y is finite, it follows that S_{-+} is of the same cardinality as S_{+-}. By the above there exists a map $h_f \in H_g$ such that $h_f(S_{+-}) = S_{-+}$ and $h_f(y) = y$ for all $y \in Y \setminus S_{+-} = S_{++}$. We obtain that $fh_f(S_{+-}) = f(h_f(S_{+-})) = f(S_{-+})$ and $fh_f(S_{++}) = f(h_f(S_{++})) = f(S_{++})$, which implies that $fh_f(Y) = Y$. Hence the map fh_f belongs to the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$, which is contained in H_g. Then $f = (fh_f)h_f^{-1}$ is in H_g as well. Thus $H_g = \mathcal{G}$. □

Theorem 7.12 allows to determine for any ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ and any finite set $Y \subset X$ whether the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ and whether $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$. Namely, $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$ if and only if Y is a union of orbits of $\mathcal{G}$ (since the set Y is finite, this means the union of a finite number of finite orbits). The stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if $Y = Y_1 \sqcup Y_2$, where Y_2 is a union of orbits of $\mathcal{G}$ (it may be empty) and Y_1 is a proper nonempty subset of a single orbit such that either Y does not contain exactly half of points in the orbit or else Y is a one-point subset of a two-point orbit.

The following concise corollary of Theorem 7.12 is much less general but it will be enough, e.g., when the topological space X is a Cantor set and the ample group $\mathcal{G}$ acts minimally on X.

Theorem 7.13. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that has no finite orbits. Suppose Y is a finite nonempty subset of X. Then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if Y is contained in a single orbit of $\mathcal{G}$.*

Proof. Since all orbits of the group $\mathcal{G}$ are infinite, the finite set Y cannot contain an entire orbit of $\mathcal{G}$ or exactly half of elements in an orbit. If Y intersects at least two different orbits of $\mathcal{G}$, then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is not a maximal subgroup of $\mathcal{G}$ due to statement (iii) of Theorem 7.12. Otherwise Y is contained in a single orbit of $\mathcal{G}$. Then $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of the group $\mathcal{G}$ due to statement (i) of Theorem 7.12. □

To treat the stabilizers of infinite closed sets, we need to develop a different approach. This approach will also apply to another class of subgroups that contains many maximal subgroups, namely, the stabilizers of partitions of X into clopen sets (see Definition 4.5).

Lemma 7.14. *Suppose $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into clopen sets. Then for any ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ is the internal direct product of the local groups $\mathcal{G}_{U_i}$, $1 \le i \le k$.*

Proof. Suppose $f \in \mathcal{G}_{U_i}$, $1 \le i \le k$. Then $f(x) = x$ for all $x \notin U_i$. Since f is an invertible map, it follows that $f(U_i) = U_i$. Hence the local groups $\mathcal{G}_{U_i}$, $1 \le i \le k$ are subgroups of the group $H = \mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. To prove that H is the internal direct product of these subgroups, we need to show that the function $\Phi : \mathcal{G}_{U_1} \times \mathcal{G}_{U_2} \times \cdots \times \mathcal{G}_{U_k} \to H$ given by $\Phi(f_1, f_2 \dots, f_k) = f_1 f_2 \dots f_k$ for all $f_i \in \mathcal{G}_{U_i}$, $1 \le i \le k$, is an isomorphism of groups.

If $g \in \mathcal{G}_{U_i}$ and $h \in \mathcal{G}_{U_j}$ for some i and j, $i \ne j$, then the maps g and h have disjoint supports. Hence $gh = hg$ due to Lemma 3.9. This fact implies that Φ is a homomorphism. Further, the map $\Phi(f_1, f_2 \dots, f_k)$ coincides with f_i on U_i for any i, $1 \le i \le k$. Besides, two elements of $\mathcal{G}_{U_i}$ are the same if and only if they coincide on U_i. These facts imply that the function Φ is one-to-one. Now take any $f \in H$. For each i, $1 \le i \le k$, consider a map $f_i : X \to X$ that coincides with f on U_i and with the identity map everywhere else. Since f is invertible and $f(U_i) = U_i$, it follows that the map f_i is invertible as well. By construction, f_i is piecewise an element of the ample group $\mathcal{G}$ and $\mathrm{supp}(f_i) \subset U_i$. Hence $f_i \in \mathcal{G}_{U_i}$. Also by construction, $\Phi(f_1, f_2 \dots, f_k) = f$. We conclude that the function Φ is onto. Thus Φ is an isomorphism. ☐

Lemma 7.15. *Let $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ be a partition of X into clopen sets at most one of which consists of a single point. Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a subgroup of $\mathcal{G}$ such that $\mathrm{St}^\bullet_{\mathcal{G}}(U_1, U_2, \dots, U_k) \subset H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Then H contains a local group $\mathcal{G}_U$ if and only if $U \subset U_i$ for some i, $1 \le i \le k$.*

Proof. Lemma 7.14 implies that the individual stabilizer $\mathrm{St}^\bullet_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ contains local groups $\mathcal{G}_{U_1}, \mathcal{G}_{U_2}, \dots, \mathcal{G}_{U_k}$. If a clopen set U is contained in U_i for some i, $1 \le i \le k$ then, clearly, $\mathcal{G}_U \subset \mathcal{G}_{U_i}$. Consequently, $\mathcal{G}_U \subset H$. Now assume that a clopen set U is not contained in a single element of the partition $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$. Then there are two different elements V_1 and V_2 of the partition such that the intersections $U \cap V_1$ and $U \cap V_2$ are not empty. By the assumption of the lemma, at least one of the sets V_1 and V_2 contains more than one point. We may assume without loss of generality that V_1 contains more than one point. Take any point $x \in U \cap V_1$. Let y be any point of V_1 different from x. Since the topological space X is totally disconnected, the point x has a clopen neighborhood V_0 that does not contain y. Further, minimality of the action of the group $\mathcal{G}$ on X implies that the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in X. In particular, this orbit has a point in the clopen set $U \cap V_2$. Hence there exists $g \in \mathcal{G}$ such that $g(x) \in U \cap V_2$. Let $W = U \cap V_1 \cap V_0 \cap g^{-1}(U \cap V_2)$. Then W is a clopen neighborhood of x. By construction, $W \subset U \cap V_1$ and $g(W) \subset U \cap V_2$, which implies that the sets W and $g(W)$ are disjoint. Therefore the generalized 2-cycle $\delta_{W;g}$ is defined. Since $g \in \mathcal{G}$, it follows that $\delta_{W;g} \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. Note that the support of the map $\delta_{W;g}$, which is $W \cup g(W)$, is contained in U. Hence $\delta_{W;g}$ belongs to the local group $\mathcal{G}_U$. On the other hand, the support of $\delta_{W;g}$ does not contain the point y since $y \in V_1 \setminus V_0$. Therefore $\delta_{W;g}(y) = y \in V_1$ while $\delta_{W;g}(x) = g(x) \in V_2$. Hence the image $\delta_{W;g}(V_1)$ intersects both V_1 and V_2 so that it cannot be an element of the partition $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$. As a consequence, $\delta_{W;g}$ does not belong to the collective stabilizer $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. In particular, $\delta_{W;g} \notin H$. We conclude that the local group $\mathcal{G}_U$ is not contained in H. ☐

Notice that the assumption of Lemma 7.15 that at most one element of the partition of X consists of a single point is essential. A one-point set $\{x\}$ is clopen if x is an isolated point of X. If $\{x\}$ and $\{y\}$ are two distinct elements of the partition, then the local group $\mathcal{G}_{\{x,y\}}$ is contained in the stabilizer of the partition.

Lemma 7.16. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose that $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ is a partition of X into nonempty clopen sets. Then $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k) \ne \mathcal{G}$ unless this is the partition into points or the trivial partition consisting of only one set.*

Proof. The partition of X into points and the trivial partition are preserved by any invertible transformation of X. In particular, they are preserved by all elements of the group $\mathcal{G}$. Conversely, assume that $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k) = \mathcal{G}$. First consider the case when each element of the partition consists of more than one point. In this case Lemma 7.15 implies that any local group $\mathcal{G}_U$ is contained in the stabilizer $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ if and only if the clopen set U is contained in a single element of the partition. Since $\mathcal{G}$ can be represented as the local group $\mathcal{G}_X$, it follows that X is an element of the partition. As all elements of the partition are nonempty sets, we obtain that X is the only element so that the partition is trivial.

Now consider the case when some element of the partition consists of a single point x_0. Minimality of the action of the group $\mathcal{G}$ on X implies that the orbit $\mathrm{Orb}_{\mathcal{G}}(x_0)$ is dense in X. That is, it has a point in any nonempty open set. In particular, the orbit has a point in each element of our partition. Hence for any i, $1 \le i \le k$ there exists $g_i \in \mathcal{G}$ such that $g_i(x_0) \in U_i$. By assumption, g_i belongs to the stabilizer of the partition. Therefore it should map $\{x_0\}$ onto another element of the partition. It follows that $U_i = \{g_i(x_0)\}$. We obtain that X is a finite set and $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ is the partition into points. ☐

Proposition 7.17. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group with Property E. Suppose H is a subgroup of $\mathcal{G}$ that acts minimally on X and contains a local group $\mathcal{G}_U$ for some clopen set U consisting of more than one point. Then there exists a partition of X into clopen sets, $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$, such that each U_i consists of more than one point and $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k) \subset H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Moreover, the partition is unique and the induced action of H on the set $\{U_1, U_2, \dots, U_k\}$ is transitive.*

Proof. Let $\mathcal{L}_H$ be the set of all clopen sets $V \subset X$ such that the subgroup H contains the local group $\mathcal{G}_V$. Note that $\mathcal{G}_{V_1} \subset \mathcal{G}_{V_2}$ whenever $V_1 \subset V_2$. Hence any clopen set contained in an element of $\mathcal{L}_H$ is itself an element of $\mathcal{L}_H$. Since the group $\mathcal{G}$ has Property E, for any intersecting clopen sets V_1 and V_2 the local group $\mathcal{G}_{V_1 \cup V_2}$ is generated by the union $\mathcal{G}_{V_1} \cup \mathcal{G}_{V_2}$. Therefore $V_1 \cup V_2 \in \mathcal{L}_H$ whenever $V_1, V_2 \in \mathcal{L}_H$ and $V_1 \cap V_2 \neq \emptyset$. According to Lemma 3.11, $\mathcal{G}_{g(V)} = g\mathcal{G}_V g^{-1}$ for any clopen set V and any $g \in \mathcal{G}$. It follows that $h(V) \in \mathcal{L}_H$ for all $h \in H$ whenever $V \in \mathcal{L}_H$.

By assumption, some clopen set U consisting of more than one point is an element of $\mathcal{L}_H$. Minimality of the action of the group H on X implies that any orbit $\mathrm{Orb}_H(x)$ of H is dense in X. In particular, it has a point in U. That is, $h(x) \in U$ for some $h \in H$. Then $h^{-1}(U)$ is an element of $\mathcal{L}_H$ containing the point x. We conclude that elements of $\mathcal{L}_H$ cover X. Since the topological space X is compact, there is a finite subcover. Let $\{U_1, U_2, \dots, U_k\}$ be a finite subcover with the least possible number of elements. This choice implies that all elements of the subcover are nonempty sets and the union of any two or more of them is not an element of $\mathcal{L}_H$. Then it follows from the above that any two elements of the subcover are disjoint. Hence our subcover is actually a partition: $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$. Observe that this is not the partition into points as otherwise U would be the union of two or more elements of the partition. We claim that any set $V \in \mathcal{L}_H$ is contained in a single element of the partition. Assume the contrary, that is, V intersects U_i and U_j, where $i \neq j$. Then the unions $V \cup U_i$ and $V \cup U_j$ are elements of $\mathcal{L}_H$. These unions are not disjoint as they both contain V. Therefore their union $V \cup U_i \cup U_j$ belongs to $\mathcal{L}_H$ as well. Since $U_i \cup U_j$ is a clopen subset of $V \cup U_i \cup U_j$, we obtain that $U_i \cup U_j \in \mathcal{L}_H$, which yields a contradiction.

Take any $h \in H$. By the above the sets $h(U_1), h(U_2), \dots, h(U_k)$ belong to $\mathcal{L}_H$. Therefore each of them is contained in a single element of the partition $\{U_1, U_2, \dots, U_k\}$. Since h is an invertible map, the sets $h(U_1), h(U_2), \dots, h(U_k)$ themselves form a partition of X. Hence we have a finite partition of X into nonempty sets each element of which is contained in a single element of another partition of X into nonempty sets. Since both partitions consist of the same number of sets, this is possible only if both partitions are the same. In other words, h maps the sets $U_1, U_2, \dots, U_k$ onto one another. We conclude that the subgroup H is contained in $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, the stabilizer of the partition. Besides, H contains the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Indeed, it follows from Lemma 7.14 that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ is generated by the local groups $\mathcal{G}_{U_i}$, $1 \le i \le k$, which are contained in H.

Since $H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, the group H acts on the set $\{U_1, U_2, \dots, U_k\}$. Take any orbit $\mathcal{O}$ of this action. Then the union of all sets in $\mathcal{O}$ is invariant under the action of H on X. Note that this union is a nonempty clopen set. Minimality of the latter action implies that the union coincides with X. Hence $\mathcal{O} = \{U_1, U_2, \dots, U_k\}$ so that the action of H on $\{U_1, U_2, \dots, U_k\}$ is transitive. In other words, all elements of the partition can be mapped onto one another by elements of H. As a consequence, all elements of the partition are of the same cardinality. Since $\{U_1, U_2, \dots, U_k\}$ is not the partition into points, it follows that each U_i consists of more than one point.

Let $\{V_1, V_2, \dots, V_\ell\}$ be an arbitrary partition of X into clopen sets such that each V_j consists of more than one point and $\mathrm{St}^{\bullet}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell) \subset H \subset \mathrm{St}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell)$. Lemma 7.14 implies that the sets $V_1, V_2, \dots, V_\ell$ belong to $\mathcal{L}_H$, just as the sets $U_1, U_2, \dots, U_k$ do. Note that the group $\mathcal{G}$ acts minimally on X since its subgroup H acts minimally. Hence we can apply Lemma 7.15. It follows from the lemma that each element of either of the partitions $\{U_1, U_2, \dots, U_k\}$ and $\{V_1, V_2, \dots, V_\ell\}$ is contained in a single element of the other partition. Since both partitions consist of nonempty sets, this is possible only if both partitions are the same. $\square$

Let us discuss the formulation of Proposition 7.17, specifically, why all clopen sets involved should consist of more than one point. If x is an isolated point of the topological space X, then $\{x\}$ is a nonempty clopen set but the associated local group $\mathcal{G}_{\{x\}}$ is trivial. Hence the condition that a clopen set U contains more than one point is necessary for the local group $\mathcal{G}_U$ to be nontrivial. In the case when the ample group $\mathcal{G}$ acts minimally on X, one can show that this condition is also sufficient. Further, if X is a finite set and $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ is the partition into points, then $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ is a trivial group while $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k) = \mathcal{G}$. The requirement that each U_i consists of more than one point rules out this degenerate case. Without it we would not be able to show uniqueness of the partition. Besides, the requirement allows to use Proposition 7.17 in conjunction with Lemma 7.15.

Proposition 7.17 is going to be crucial for the proofs of all subsequent results on maximal subgroups. The downside of this is that all those results will require the ample group to have Property E. We shall discuss this property and provide examples of ample groups with it in Section 9 below.

Our next result concerns the stabilizers of closed sets that are not open.

Theorem 7.18. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property E. Suppose $Y \subset X$ is a closed set that is not open. Then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a* *maximal subgroup of $\mathcal{G}$ if and only if it acts minimally when restricted to Y, in which case Y is necessarily nowhere dense in X.*

Proof. If the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$, then the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal due to Theorem 7.6. Since the set Y is not clopen, Lemma 7.7 implies that it is nowhere dense in X.

Now assume that $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y. Just as above, this implies that Y is nowhere dense in X. Also, since the closed set Y is not open, it is different from the empty set and X. Then $\mathrm{St}_{\mathcal{G}}(Y) \neq \mathcal{G}$ due to Lemma 7.2. To prove that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$, we are going to show that for any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(Y)$ the group $H_g = \langle \mathrm{St}_{\mathcal{G}}(Y) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(Y)$ and g coincides with $\mathcal{G}$. By Lemma 7.9, the group H_g has the same orbits as $\mathcal{G}$. As a consequence, H_g acts minimally on X. To apply Proposition 7.17 to the group H_g, we need it to contain a local group $\mathcal{G}_U$ for some clopen set U containing more than one point. If $U \subset X \setminus Y$ then any map in $\mathcal{G}_U$ fixes all points of Y so that $\mathcal{G}_U \subset \mathrm{St}^{\star}_{\mathcal{G}}(Y) \subset \mathrm{St}_{\mathcal{G}}(Y) \subset H_g$. Note that the set $X \setminus Y$ is open but not closed. Therefore it is infinite. Since the topological space X is totally disconnected and compact, it follows that any point $x \in X \setminus Y$ has a clopen neighborhood $V_x \subset X \setminus Y$. Take any distinct points $x, y \in X \setminus Y$. Then $V_x \cup V_y$ is a clopen set consisting of more than one point. Besides, $V_x \cup V_y \subset X \setminus Y$ so that $\mathcal{G}_{V_x \cup V_y} \subset H_g$. Now Proposition 7.17 implies that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k) \subset H_g \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, where $\mathcal{P} = \{U_1, U_2, \dots, U_k\}$ is a partition of X into clopen sets, each consisting of more than one point. Since the set Y is nowhere dense in X, its complement intersects any nonempty open set. In particular, for any i, $1 \leq i \leq k$ we can find a point $x_i \in U_i$ that does not belong to Y. Let $V = V_{x_1} \cup V_{x_2} \cup \dots \cup V_{x_k}$. Then V is a clopen subset of $X \setminus Y$, which implies that $\mathcal{G}_V \subset H_g$. Lemma 7.15 further implies that V is contained in a single element of the partition $\mathcal{P}$. However V was constructed so as to intersect every element of $\mathcal{P}$. We conclude that $\mathcal{P}$ consists of a single element, $\mathcal{P} = \{X\}$. Since $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, we obtain that $H_g = \mathcal{G}$. $\square$

If a closed set $Y \subset X$ is clopen then for any ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ that acts minimally on X, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y (due to Lemma 7.8). If Y is finite then $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally on it if and only if Y is contained in a single orbit of $\mathcal{G}$. If the closed set Y is neither clopen nor finite, it is not clear whether minimality of the action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is at all possible. In other words, it is not clear whether Theorem 7.18 yields any new examples of maximal subgroups compared with Theorem 7.12. We address this question in Section 8 below. In view of Lemma 5.10, the assumptions of Theorem 7.18 can be satisfied only in the case when X is a Cantor set. As it turns out, there are plenty of new examples in that case (see Proposition 8.2 below).

The next three theorems contain our main results on the stabilizers of partitions of X into clopen sets. Theorems 7.19 and 7.20 tell when such stabilizers are maximal subgroups of the corresponding ample groups (Theorem 7.19 for partitions into two sets, Theorem 7.20 for partitions into three or more sets). Theorem 7.21 provides a characterization of those maximal subgroups of ample groups that are the stabilizers of partitions into clopen sets (just as Theorem 7.6 provides a characterization of those maximal subgroups that are the stabilizers of closed sets). Theorem 7.19 also treats the stabilizers of clopen sets.

Theorem 7.19. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property E. Suppose U is a clopen set different from the empty set and X. Then $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ is a maximal subgroup of $\mathcal{G}$ unless both U and $X \setminus U$ consist of a single point, in which case $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) = \mathcal{G}$. If U cannot be mapped onto $X \setminus U$ by an element of $\mathcal{G}$ then $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$; otherwise $\mathrm{St}_{\mathcal{G}}(U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$.*

Proof. For any invertible map $f : X \to X$ we have $f(U) = U$ if and only if $f(X \setminus U) = X \setminus U$. It follows that $\mathrm{St}_{\mathcal{G}}(U)$ coincides with another set stabilizer $\mathrm{St}_{\mathcal{G}}(X \setminus U)$ as well as with the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U)$. The stabilizer $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ of the partition $X = U \sqcup (X \setminus U)$ acts on the set $\{U, X \setminus U\}$ and $\mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U)$ is the kernel of this action. If U cannot be mapped onto $X \setminus U$ by an element of $\mathcal{G}$, then the action is trivial so that $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) = \mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U) = \mathrm{St}_{\mathcal{G}}(U)$. Otherwise $f(U) = X \setminus U$ for some $f \in \mathcal{G}$. Then also $f(X \setminus U) = X \setminus f(U) = U$ so that $f \in \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$. Hence the action of the group $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ on $\{U, X \setminus U\}$ is nontrivial, which implies that $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$.

If both U and $X \setminus U$ consist of a single point, then $X = U \sqcup (X \setminus U)$ is the partition into points, which implies that $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) = \mathcal{G}$. Otherwise $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) \neq \mathcal{G}$ due to Lemma 7.16. To prove that $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ is a maximal subgroup of $\mathcal{G}$ in the latter case, we are going to show that for any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ the group $H_g = \langle \mathrm{St}_{\mathcal{G}}(U, X \setminus U) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ and g coincides with $\mathcal{G}$. Since the group $\mathcal{G}$ acts minimally on X, it follows from Lemma 7.8 that the stabilizer $\mathrm{St}_{\mathcal{G}}(U)$ acts minimally when restricted to U. Then Lemma 7.9 implies that the group $\langle \mathrm{St}_{\mathcal{G}}(U) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(U)$ and g has the same orbits as $\mathcal{G}$. Since the group H_g contains $\langle \mathrm{St}_{\mathcal{G}}(U) \cup \{g\} \rangle$, it also has the same orbits as $\mathcal{G}$. As a consequence, H_g acts minimally on X. Further, Lemma 7.14 implies that the stabilizer $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ contains the local groups $\mathcal{G}_U$ and $\mathcal{G}_{X \setminus U}$. Hence H_g contains them as well. Recall that we consider the case when at least one of the sets U and $X \setminus U$ consists of more than one point. Now Proposition 7.17 implies that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k) \subset H_g \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, where $\mathcal{P} = \{U_1, U_2, \dots, U_k\}$ is a partition of X into clopen sets, each consisting of more than one point. By Lemma 7.15, any local group $\mathcal{G}_V$ is a subgroup of H_g if and only if $V \subset U_i$ for some i, $1 \leq i \leq k$. In particular, each of the sets U and $X \setminus U$ is contained in a single element of the partition $\mathcal{P}$. This leaves only two possibilities, $\mathcal{P} = \{U, X \setminus U\}$ or $\mathcal{P} = \{X\}$. Since H_g is not contained in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$, we conclude that $\mathcal{P} = \{X\}$. As $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, we obtain that $H_g = \mathcal{G}$. ☐

Theorem 7.20. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property E. Suppose $X = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ is a partition of X into at least three clopen sets, each consisting of more than one point. Then $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, the stabilizer of the partition, is a maximal subgroup of $\mathcal{G}$ if and only if its induced action on the set $\{U_1, U_2, \dots, U_k\}$ is transitive.*

Proof. Let $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Since the partition $\mathcal{P} = \{U_1, U_2, \dots, U_k\}$ is neither trivial nor the partition into points, Lemma 7.16 implies that $H \neq \mathcal{G}$. First we consider the case when the induced action of the group H on $\mathcal{P}$ is not transitive. Take any orbit $\mathcal{O}$ of this action and let U be the union of all sets in $\mathcal{O}$. Then U is a clopen set invariant under the action of H on X. It follows that $H \subset \mathrm{St}_{\mathcal{G}}(U)$. Note that the orbit $\mathcal{O}$ is different from the empty set and $\mathcal{P}$. Therefore U is different from the empty set and X. Then $\mathrm{St}_{\mathcal{G}}(U) \neq \mathcal{G}$ due to Lemma 7.2. By Lemma 4.4, the pointwise stabilizer $\mathrm{St}^{\star}_{\mathcal{G}}(U)$ and the rigid stabilizer $\mathrm{RiSt}_{\mathcal{G}}(U)$ are subgroups of $\mathrm{St}_{\mathcal{G}}(U)$. It is easy to see that $\mathrm{St}^{\star}_{\mathcal{G}}(U) = \mathcal{G}_{X\setminus U}$ and $\mathrm{RiSt}_{\mathcal{G}}(U) = \mathcal{G}_U$. On the other hand, the local groups $\mathcal{G}_U$ and $\mathcal{G}_{X\setminus U}$ cannot both be subgroups of H. Indeed, otherwise it would follow from Lemma 7.15 that each of the sets U and $X \setminus U$ is contained in a single element of the partition $\mathcal{P}$, which is not possible as $k \geq 3$. We conclude that $H \neq \mathrm{St}_{\mathcal{G}}(U)$. Hence H is not a maximal subgroup of $\mathcal{G}$.

Now consider the case when the induced action of H on the partition $\mathcal{P}$ is transitive. Let us show that in this case the group H has the same orbits as $\mathcal{G}$. Take any $x \in X$ and let V be the element of $\mathcal{P}$ containing x. Since H acts transitively on $\mathcal{P}$, for any i, $1 \leq i \leq k$ there exists $h_i \in H$ such that $h_i(V) = U_i$. Let $y_i = h_i(x)$. By Lemma 7.9, the orbit of the point y_i under the action of the stabilizer $\mathrm{St}_{\mathcal{G}}(U_i)$ coincides with $\mathrm{Orb}_{\mathcal{G}}(y_i) \cap U_i$. Just as above, Lemma 4.4 implies that the local group $\mathcal{G}_{U_i}$ is a subgroup of $\mathrm{St}_{\mathcal{G}}(U_i)$. Given $g \in \mathrm{St}_{\mathcal{G}}(U_i)$, let h be a map that coincides with g on U_i and with the identity map on $X \setminus U_i$. Then $h \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. By construction, $h \in \mathcal{G}_{U_i}$. It follows that the orbit of y_i under the action of $\mathrm{St}_{\mathcal{G}}(U_i)$ coincides with its orbit under the action of $\mathcal{G}_{U_i}$. Further, Lemma 7.14 implies that the local group $\mathcal{G}_{U_i}$ is a subgroup of the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, which in turn is a subgroup of H due to Lemma 4.6. Hence the orbit of y_i under the action of $\mathcal{G}_{U_i}$ is contained in $\mathrm{Orb}_H(y_i)$. Therefore $\mathrm{Orb}_{\mathcal{G}}(y_i) \cap U_i \subset \mathrm{Orb}_H(y_i)$. Clearly, $\mathrm{Orb}_H(y_i) = \mathrm{Orb}_H(x)$ and $\mathrm{Orb}_{\mathcal{G}}(y_i) = \mathrm{Orb}_{\mathcal{G}}(x)$. We conclude that $\mathrm{Orb}_{\mathcal{G}}(x) \cap U_i \subset \mathrm{Orb}_H(x)$ for all i, $1 \leq i \leq k$, which implies that $\mathrm{Orb}_H(x) = \mathrm{Orb}_{\mathcal{G}}(x)$.

To prove that H is a maximal subgroup of $\mathcal{G}$, we need to show that for any $g \in \mathcal{G} \setminus H$ the group $H_g = \langle H \cup \{g\} \rangle$ generated by H and g coincides with $\mathcal{G}$. Since the group H has the same orbits as $\mathcal{G}$, the same is true for H_g. As a consequence, H_g acts minimally on X. Since the group H contains the local groups $\mathcal{G}_{U_i}$, $1 \leq i \leq k$, so does H_g. Now Proposition 7.17 implies that $\mathrm{St}^{\bullet}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell) \subset H_g \subset \mathrm{St}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell)$, where $\mathcal{P}_g = \{V_1, V_2, \dots, V_\ell\}$ is a partition of X into clopen sets, each consisting of more than one point. By Lemma 7.15, any local group $\mathcal{G}_U$ is a subgroup of H_g if and only if $U \subset V_j$ for some j, $1 \leq j \leq \ell$. In particular, each element of the partition $\mathcal{P}$ is contained in a single element of the partition $\mathcal{P}_g$. It follows that each element of $\mathcal{P}_g$ is the union of one or more elements of $\mathcal{P}$. Note that $\mathcal{P}_g \neq \mathcal{P}$ since the group H_g is not contained in H. Hence there exists j, $1 \leq j \leq \ell$ such that the set V_j contains sets U_{i_1} and U_{i_2}, where $i_1 \neq i_2$. Take any i, $1 \leq i \leq k$, different from i_1 and i_2. Since the group H acts transitively on $\mathcal{P}$, we have $h(U_{i_2}) = U_i$ for some $h \in H$. Note that U_{i_2} and U_i are disjoint clopen sets. Hence the generalized 2-cycle $f = \delta_{U_{i_2};h}$ is defined. Since $h \in \mathcal{G}$, it follows that $f \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. We have $f(U_{i_2}) = U_i$, $f(U_i) = U_{i_2}$, and $f(x) = x$ for any x not in U_{i_2} or U_i. Therefore f preserves the partition $\mathcal{P}$ so that $f \in H$. Then $f \in H_g$ so that f preserves the partition $\mathcal{P}_g$ as well. In particular, $f(V_j) \in \mathcal{P}_g$. Since $f(U_{i_1}) = U_{i_1}$ and $f(U_{i_2}) = U_i$, it follows that $f(V_j) = V_j$ and $U_i \subset V_j$. As i was chosen arbitrarily, we conclude that $V_j = X$. Thus the partition $\mathcal{P}_g$ is trivial, $\mathcal{P}_g = \{X\}$. Since $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, we obtain that $H_g = \mathcal{G}$. $\square$

Theorem 7.21. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group with Property E. Suppose H is a maximal subgroup of $\mathcal{G}$ that acts minimally on X and contains a local group $\mathcal{G}_U$ for some clopen set U containing more than one point. Then $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, where $X =$*

$U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ *is a partition of* X *into clopen sets, each containing more than one point. Moreover, the partition is unique, it consists of at least two sets, and the induced action of* H *on the set* $\{U_1, U_2, \ldots, U_k\}$ *is transitive.*

Proof. By Proposition 7.17, there exists a partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ such that each U_i is a clopen set consisting of more than one point and $\mathrm{St}^\bullet_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \subset H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$. Moreover, the partition is unique and the induced action of H on the set $\{U_1, U_2, \ldots, U_k\}$ is transitive. The partition cannot be trivial (with $k = 1$ and $U_1 = X$) as $\mathrm{St}^\bullet_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$ while $H \neq \mathcal{G}$. Hence $k \geq 2$. Besides, this is clearly not the partition into points. Then $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \neq \mathcal{G}$ due to Lemma 7.16. Since $H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ and H is a maximal subgroup of $\mathcal{G}$, it follows that $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$. □

8 Nowhere dense closed sets

In this section we present a construction of a closed, nowhere dense subset Y of a topological space X and a locally finite group H of homeomorphisms of X that leave Y invariant. The closed set Y is a Cantor set and the group H acts minimally when restricted to Y. In the case when X is a Cantor set, the construction helps, given an ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ that acts minimally on X, to produce uncountably many infinite, nowhere dense closed sets Y such that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally on Y. By Theorem 7.18, all those stabilizers are maximal subgroups of $\mathcal{G}$ provided that the group $\mathcal{G}$ has Property E. Besides, it follows from Lemma 7.4 that all those stabilizers are different from one another and from the stabilizers of other closed sets.

The construction is general and we do not assume the topological space X to be a Cantor set. Suppose $U_0 \supset U_1 \supset U_2 \supset \ldots$ is a nested sequence of nonempty clopen subsets of X and x_0 is a common point of all these sets. Further suppose that for any $n \geq 1$ we have a homeomorphism $g_n : X \to X$ such that the set $g_n(U_n)$ is disjoint from U_n and, moreover, the union $U_n \cup g_n(U_n)$ is a proper subset of U_{n-1}. Let H be a subgroup of $\mathrm{Homeo}(X)$ generated by the generalized 2-cycles $\delta_{U_n;g_n}$, $n = 1, 2, \ldots$ Let $Y = \overline{\mathrm{Orb}_H(x_0)}$, the closure of the orbit of the point x_0 under the natural action of the group H on X. Then Y is a closed set invariant under the action of H. Assuming all maps g_n, $n \geq 1$ are taken from a group $G \subset \mathrm{Homeo}(X)$, the group H is a subgroup of the ample group $\mathcal{G} = \mathsf{F}(G)$ so that $H \subset \mathrm{St}_{\mathcal{G}}(Y)$. For any $n \geq 1$, we let $f_n^{(0)} = \mathrm{id}_X$ and $f_n^{(1)} = \delta_{U_n;g_n}$. Note that $f_n^{(0)}(U_n) = U_n$ and $f_n^{(1)}(U_n) = g_n(U_n)$ are subsets of U_{n-1}. Hence for any infinite string $\xi_1\xi_2\xi_3\ldots$ of 0s and 1s we have a nested sequence of clopen sets $U_0 \supset f_1^{(\xi_1)}(U_1) \supset f_1^{(\xi_1)} f_2^{(\xi_2)}(U_2) \supset \ldots$. The intersection of these sets is nonempty.

Lemma 8.1. *In terms of the above construction, suppose that for any infinite string* $\xi_1\xi_2\xi_3\ldots$ *of 0s and 1s the intersection of the nested sets* $U_0 \supset f_1^{(\xi_1)}(U_1) \supset f_1^{(\xi_1)} f_2^{(\xi_2)}(U_2) \supset \ldots$ *consists of a single point. Then* Y *is a Cantor set that is nowhere dense in* X *and the group* H *acts minimally when restricted to* Y*. Besides,* H *is locally finite, which means that any finitely generated subgroup of* H *is finite.*

Proof. We keep the same notation as used in the construction. First let us describe an abstract model behind the construction. Let $\Xi = \{0,1\}^{\mathbb{N}}$. We regard any element $\xi \in \Xi$ as an infinite string $\xi = \xi_1\xi_2\xi_3\dots$, where each $\xi_i \in \{0,1\}$. We endow Ξ with the product topology (assuming the discrete topology on $\{0,1\}$). Then Ξ is a Cantor set. Let $\{0,1\}^*$ be the set of all finite strings of 0s and 1s (including the empty string $\varnothing$). For any $w \in \{0,1\}^*$ we denote by C_w the set of all infinite strings in Ξ that begin with w. Sets of the form C_w are called *cylinders*, they form a base of the topology on Ξ.

For any $n \geq 1$ let γ_n be a transformation of Ξ that changes only the n-th character in any infinite string that begins with (at least) $n-1$ consecutive zeros, and fixes all the other elements of Ξ. The transformation γ_n is clearly an involution. Besides, it is continuous. Hence $\gamma_n \in \mathrm{Homeo}(\Xi)$. Let Γ be the subgroup of $\mathrm{Homeo}(\Xi)$ generated by $\gamma_1, \gamma_2, \gamma_3, \dots$.

For any $\xi = \xi_1\xi_2\xi_3\dots \in \Xi$ and integer $n \geq 0$ let $Z(\xi,n)$ denote a string $\zeta = \zeta_1\zeta_2\zeta_3\dots$ that begins with n zeros and coincides with the string ξ afterwards: $\zeta_k = 0$ if $k \leq n$ and $\zeta_k = \xi_k$ if $k > n$. Clearly, $Z(\xi,0) = \xi$. For any $n \geq 1$ we have $Z(\xi,n) = Z(\xi,n-1)$ if $\xi_n = 0$ and $Z(\xi,n) = \gamma_n(Z(\xi,n-1))$ if $\xi_n = 1$. It follows that the strings $Z(\xi,0), Z(\xi,1), Z(\xi,2), \dots$ all belong to the orbit $\mathrm{Orb}_\Gamma(\xi)$. If two strings $\xi, \xi' \in \Xi$ coincide up to finitely many characters, then $Z(\xi,n) = Z(\xi',n)$ for all sufficiently large n, which implies that ξ and ξ' are in the same orbit of the group Γ. We conclude that any orbit of Γ has an element in every cylinder C_w, $w \in \{0,1\}^*$. As a consequence, any orbit of Γ is dense in Ξ. Therefore the group Γ acts minimally on Ξ.

Now we are going to show that the actual construction of the set Y and the group H agrees with our abstract model (respectively Ξ and Γ). First we define a map $\alpha : \Xi \to Y$. Given $\xi = \xi_1\xi_2\xi_3\dots \in \Xi$, consider a nested sequence of clopen sets $U_0 \supset f_1^{(\xi_1)}(U_1) \supset f_1^{(\xi_1)}f_2^{(\xi_2)}(U_2) \supset \dots$. By assumption, the intersection of all these sets consists of a single point, and we define $\alpha(\xi)$ to be that point. Let us show that $\alpha(\xi) \in Y$. Since the point x_0 belongs to each of the sets $U_0, U_1, U_2, \dots$, it follows that $f_1^{(\xi_1)}f_2^{(\xi_2)}\dots f_n^{(\xi_n)}(x_0) \in f_1^{(\xi_1)}f_2^{(\xi_2)}\dots f_n^{(\xi_n)}(U_n)$ for $n = 1,2,\dots$. This implies that every limit point of the sequence $x_0, f_1^{(\xi_1)}(x_0), f_1^{(\xi_1)}f_2^{(\xi_2)}(x_0), \dots$ belongs to the intersection of the nested clopen sets $U_0, f_1^{(\xi_1)}(U_1), f_1^{(\xi_1)}f_2^{(\xi_2)}(U_2), \dots$. As $\alpha(\xi)$ is the only point in the intersection, we conclude that the sequence converges to $\alpha(\xi)$. Since all points in the sequence belong to the orbit $\mathrm{Orb}_H(x_0)$, we obtain that $\alpha(\xi) \in \overline{\mathrm{Orb}_H(x_0)} = Y$.

For any finite string $w = \xi_1\xi_2\dots\xi_n$ of 0s and 1s consisting of $n \geq 1$ characters, we let $f^{(w)} = f_1^{(\xi_1)}f_2^{(\xi_2)}\dots f_n^{(\xi_n)}$ and $V^{(w)} = f^{(w)}(U_n)$. Also, we let $f^{(\varnothing)} = \mathrm{id}_X$ and $V^{(\varnothing)} = U_0$. By construction, $\alpha(C_w) \subset V^{(w)}$ for all $w \in \{0,1\}^*$. For any strings $w, u \in \{0,1\}^*$ let wu denote their concatenation. Notice that $C_{wu} \subset C_w$ and $V^{(wu)} \subset V^{(w)}$.

Let $w \in \{0,1\}^*$ and n be the number of characters in the string w. We have $V^{(w)} = f^{(w)}(U_n)$, $V^{(w0)} = f^{(w)}(U_{n+1})$ and $V^{(w1)} = f^{(w)}\big(f^{(1)}(U_{n+1})\big)$. By construction, U_n is the disjoint union of three nonempty clopen sets U_{n+1}, $f^{(1)}(U_{n+1})$ and $U_n \setminus \big(U_{n+1} \cup f^{(1)}(U_{n+1})\big)$. It follows that $V^{(w)}$ is the disjoint union of three nonempty clopen sets $V^{(w0)}$, $V^{(w1)}$ and $V^{(w)} \setminus \big(V^{(w0)} \cup V^{(w1)}\big)$. Note that any two strings in $\{0,1\}^*$ neither of which is a beginning of the other can be written as $w0u$ and $w1u'$, where $w, u, u' \in \{0,1\}^*$. Since $V^{(w0u)} \subset V^{(w0)}$ and $V^{(w1u')} \subset V^{(w1)}$, we obtain that the set $V^{(w0u)}$ is disjoint from $V^{(w1u')}$. In particular, the sets $V^{(w)}$ and $V^{(w')}$ are disjoint if w and w' are two different strings with the same number of characters.

Next we prove that the map α is one-to-one. Given two infinite strings $\xi, \xi' \in \Xi$, let w be their longest common beginning. If $\xi \neq \xi'$ then w is a finite string. Moreover, one of the strings ξ and ξ' belongs to C_{w0} while the other is in C_{w1}. Hence one of the points $\alpha(\xi)$ and $\alpha(\xi')$ belongs to $V^{(w0)}$ while the other is in $V^{(w1)}$. By the above the sets $V^{(w0)}$ and $V^{(w1)}$ are disjoint. As a consequence, $\alpha(\xi) \neq \alpha(\xi')$.

Given $\xi = \xi_1\xi_2\xi_3 \ldots \in \Xi$, let us show that every open neighborhood of the point $\alpha(\xi)$ contains a clopen set of the form $V^{(w)}$, where the finite string $w \in \{0,1\}^*$ is a beginning of ξ. By construction, $V^{(\xi_1)} \supset V^{(\xi_1\xi_2)} \supset V^{(\xi_1\xi_2\xi_3)} \supset \ldots$ and the intersection of these clopen sets is $\{\alpha(\xi)\}$. Hence $X \setminus V^{(\xi_1)} \subset X \setminus V^{(\xi_1\xi_2)} \subset X \setminus V^{(\xi_1\xi_2\xi_3)} \subset \ldots$ and the union of these complements is $X \setminus \{\alpha(\xi)\}$. Adding an arbitrary open neighborhood U of $\alpha(\xi)$ to the complements, we obtain an open cover of X. Since X is compact, there is a finite subcover. It further follows that X is covered by two sets U and $X \setminus V^{(\xi_1\xi_2\ldots\xi_n)}$ provided that n is large enough. Equivalently, $V^{(\xi_1\xi_2\ldots\xi_n)} \subset U$ if n is large enough.

Next we prove that the map α is continuous. Let $\xi \in \Xi$ and U be an open neighborhood of the point $\alpha(\xi)$. We need to show that the preimage $\alpha^{-1}(U)$ contains an open neighborhood of ξ. By the above U contains a set $V^{(w)}$ for some string w that ξ begins with. Then the cylinder C_w is a clopen neighborhood of ξ. It is contained in $\alpha^{-1}(U)$ since $\alpha(C_w) \subset V^{(w)}$.

For any $n \geq 1$ we define a map $\tilde{\gamma}_n : \{0,1\}^* \to \{0,1\}^*$ as follows. If a string $w \in \{0,1\}^*$ contains at least n characters and begins with at least $n-1$ 0s, then $\tilde{\gamma}_n(w)$ is obtained from w by changing its n-th character. Otherwise $\tilde{\gamma}_n(w) = w$. By definition, if an infinite string $\xi \in \Xi$ begins with a finite string $w \in \{0,1\}^*$ then $\gamma_n(\xi)$ begins with $\tilde{\gamma}_n(w)$. Let us show that $f_n^{(1)} f^{(w)} = f^{(\tilde{\gamma}_n(w))}$ on U_{n-1} for any string w with at least n characters. Suppose $w = \xi_1\xi_2\ldots\xi_k$, where $k \geq n$. If the first $n-1$ characters of w are all 0s then $V^{(\xi_1\xi_2\ldots\xi_{n-1})} = U_{n-1}$; otherwise $V^{(\xi_1\xi_2\ldots\xi_{n-1})}$ is disjoint from U_{n-1}. In the latter case, $f_n^{(1)}$ coincides with the identity map on $V^{(\xi_1\xi_2\ldots\xi_{n-1})}$ as the support of $f_n^{(1)}$ is contained in U_{n-1}. Since $f_n^{(\xi_n)} f_{n+1}^{(\xi_{n+1})} \ldots f_k^{(\xi_k)}(U_{n-1}) = U_{n-1}$ and $f^{(\xi_1\xi_2\ldots\xi_{n-1})}(U_{n-1}) = V^{(\xi_1\xi_2\ldots\xi_{n-1})}$, we obtain that $f_n^{(1)} f^{(w)} = f^{(w)} = f^{(\tilde{\gamma}_n(w))}$ on U_{n-1}. In the former case, when $\xi_i = 0$ for all $i < n$, we have $f^{(w)} = f_n^{(\xi_n)} f_{n+1}^{(\xi_{n+1})} \ldots f_k^{(\xi_k)}$ and $f^{(\tilde{\gamma}_n(w))} = f_n^{(\xi'_n)} f_{n+1}^{(\xi_{n+1})} \ldots f_k^{(\xi_k)}$, where $\xi'_n = 1$ if $\xi_n = 0$ and $\xi'_n = 0$ if $\xi_n = 1$. Since $f_n^{(1)}$ is an involution and $f_n^{(0)} = \mathrm{id}_X$, it follows that $f_n^{(1)} f_n^{(\xi_n)} = f_n^{(\xi'_n)}$. Consequently, $f_n^{(1)} f^{(w)} = f^{(\tilde{\gamma}_n(w))}$ everywhere on X.

Next we show that $f_n^{(1)}\alpha = \alpha\gamma_n$ on Ξ for all $n \geq 1$. Take any $\xi = \xi_1\xi_2\xi_3 \ldots \in \Xi$ and let $\gamma_n(\xi) = \eta_1\eta_2\eta_3 \ldots$. By the above, $f^{(\xi_1\xi_2\ldots\xi_k)}(x_0) \to \alpha(\xi)$ and $f^{(\eta_1\eta_2\ldots\eta_k)}(x_0) \to \alpha(\gamma_n(\xi))$ as $k \to \infty$. Then $f_n^{(1)} f^{(\xi_1\xi_2\ldots\xi_k)}(x_0) \to f_n^{(1)}(\alpha(\xi))$ as $k \to \infty$. Since $x_0 \in U_{n-1}$ and $f_n^{(1)} f^{(w)} = f^{(\tilde{\gamma}_n(w))}$ on U_{n-1} for any string w with at least n characters, it follows that $f_n^{(1)} f^{(\xi_1\xi_2\ldots\xi_k)}(x_0) = f^{(\tilde{\gamma}_n(\xi_1\xi_2\ldots\xi_k))}(x_0) = f^{(\eta_1\eta_2\ldots\eta_k)}(x_0)$ for all $k \geq n$. Therefore $f_n^{(1)}(\alpha(\xi)) = \alpha(\gamma_n(\xi))$.

Next we prove that the map α is onto. For any $n \geq 1$, the equality $f_n^{(1)}\alpha = \alpha\gamma_n$ implies that $f_n^{(1)}(\alpha(\Xi)) = \alpha(\gamma_n(\Xi)) = \alpha(\Xi)$. Since the group H is generated by the maps $f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, \ldots$, it follows that $h(\alpha(\Xi)) = \alpha(\Xi)$ for all $h \in H$. Notice that $x_0 \in \alpha(\Xi)$, namely, x_0 is the image of the infinite string of all 0s. Since the set $\alpha(\Xi)$ is invariant under the action of the group H, we obtain that $\mathrm{Orb}_H(x_0) \subset \alpha(\Xi)$. Further, the set $\alpha(\Xi)$ is compact as Ξ is compact and the map α is continuous. As a consequence, $\alpha(\Xi)$ is a closed subset of X. Since $\mathrm{Orb}_H(x_0) \subset \alpha(\Xi) \subset Y$ and $Y = \overline{\mathrm{Orb}_H(x_0)}$, we conclude that $\alpha(\Xi) = Y$.

Since the map α is one-to-one and onto, it is invertible. The inverse map $\alpha^{-1} : Y \to \Xi$ is continuous since α is continuous and Ξ is compact. Hence α is a homeomorphism. As a consequence, Y is a Cantor set.

Next we derive minimality of the action of the group H on Y from minimality of the action of the group Γ on Ξ. For any closed set $Y_0 \subset Y$, the set $\alpha^{-1}(Y_0)$ is a closed subset of Ξ. Assuming Y_0 is invariant under the action of H, we have $f_n^{(1)}(Y_0) = Y_0$ for all $n \geq 1$. By the above $f_n^{(1)}\alpha = \alpha\gamma_n$ on Ξ. Therefore $\gamma_n = \alpha^{-1}f_n^{(1)}\alpha$ on Ξ. Then $\gamma_n(\alpha^{-1}(Y_0)) = \alpha^{-1}f_n^{(1)}(Y_0) = \alpha^{-1}(Y_0)$. Since the group Γ is generated by maps $\gamma_1, \gamma_2, \gamma_3, \dots$, it follows that $\gamma(\alpha^{-1}(Y_0)) = \alpha^{-1}(Y_0)$ for all $\gamma \in \Gamma$, that is, the set $\alpha^{-1}(Y_0)$ is invariant under the action of the group Γ on Ξ. Minimality of the latter action implies that $\alpha^{-1}(Y_0)$ is either Ξ or the empty set. Then Y_0 is either Y or the empty set.

Now we prove that the set Y is nowhere dense in X. Since Y is closed, it is enough to show that it has no interior points. Let $y \in Y$ and U be an open neighborhood of the point y. We have $y = \alpha(\xi)$ for some $\xi \in \Xi$. By the above U contains a set $V^{(w)}$ for some string $w \in \{0,1\}^*$ that ξ begins with. Also by the above, $V^{(w)}$ is the disjoint union of three nonempty clopen sets $V^{(w0)}$, $V^{(w1)}$ and $V^{(w)} \setminus \big(V^{(w0)} \cup V^{(w1)}\big)$. Take any string $\xi' \in \Xi$ and let w' be the beginning of ξ' that has the same number of characters as w. If $w' \neq w$ then the set $V^{(w')}$, which contains the point $\alpha(\xi')$, is disjoint from $V^{(w)}$ so that $\alpha(\xi') \notin V^{(w)}$. If $w' = w$ then $\alpha(\xi') \in V^{(w0)} \cup V^{(w1)}$. We conclude that the nonempty set $V^{(w)} \setminus \big(V^{(w0)} \cup V^{(w1)}\big)$ is disjoint from $\alpha(\Xi) = Y$. Therefore the open set U is not fully contained in Y.

It remains to prove that the group H is locally finite. For any $k \geq 1$ let $\mathcal{L}_k$ be a list of all 2^k maps of the form $f^{(w)}$, where $w \in \{0,1\}^*$ is a string of k characters. By the above the sets $f(U_k)$, $f \in \mathcal{L}_k$ are pairwise disjoint. Hence for any $\pi \in \mathsf{S}_{2^k}$ a generalized permutation $\mu[U_k; \mathcal{L}_k; \pi]$ is well defined. We denote by F_k the set of all such generalized permutations. Similarly, a generalized permutation $\mu[U_k \setminus U_{k+1}; \mathcal{L}_k; \pi]$ is defined for all $\pi \in \mathsf{S}_{2^k}$. We denote the set of all such generalized permutations by $\widetilde{F}_k$. Lemma 3.4 implies that both F_k and $\widetilde{F}_k$ are subgroups of $\mathrm{Homeo}(X)$. For any $k \geq 1$ let W_k be the union of all sets of the form $f(U_k)$, where $f \in \mathcal{L}_k$. Then $\mathrm{supp}(h) \subset W_k$ for each $h \in F_k$ and $\mathrm{supp}(h) \subset W_k \setminus W_{k+1}$ for each $h \in \widetilde{F}_k$.

By the above, $f_m^{(1)} f^{(w)} = f^{(\tilde{\gamma}_m(w))}$ on U_{m-1} for any string $w = \xi_1\xi_2\dots\xi_k \in \{0,1\}^*$ such that $k \geq m$. In particular, $f_m^{(1)} f^{(w)} = f^{(\tilde{\gamma}_m(w))}$ on U_k. Equivalently, the restriction of the map $f_m^{(1)}$ to the set $f^{(w)}(U_k)$ coincides with the map $f^{(\tilde{\gamma}_m(w))}\big(f^{(w)}\big)^{-1}$. Note that $f^{(w)}$ and $f^{(\tilde{\gamma}_m(w))}$ are both on the list $\mathcal{L}_k$. It follows that for any $k \geq m$ the restriction of the map $f_m^{(1)}$ to the set W_k coincides with an element of F_k. That is, $f_m^{(1)} = \mu[U_k; \mathcal{L}_k; \pi]$ on W_k for some $\pi \in \mathsf{S}_{2^k}$. Then $f_m^{(1)} = \mu[U_k \setminus U_{k+1}; \mathcal{L}_k; \pi]$ on $W_k \setminus W_{k+1}$, which means that the restriction of $f_m^{(1)}$ to the set $W_k \setminus W_{k+1}$ coincides with an element of $\widetilde{F}_k$. In the case when $1 \leq k < m$, the map $f_m^{(1)}$ coincides with the identity map on $W_k \setminus W_{k+1}$ as $\mathrm{supp}(f_m^{(1)}) \subset W_m \subset W_{k+1}$. Since $\mathrm{id}_X \in \widetilde{F}_k$, we obtain that the restriction of $f_m^{(1)}$ to the set $W_k \setminus W_{k+1}$ coincides with an element of $\widetilde{F}_k$ for any $k \geq 1$.

To prove that the group H is locally finite, it is enough to show that for any $n \geq 1$ the group H_n generated by $f_1^{(1)}, f_2^{(1)}, \dots, f_n^{(1)}$ is finite. Note that X is the disjoint union of the sets $X \setminus W_1, W_1 \setminus W_2, \dots, W_{n-1} \setminus W_n$ and W_n. Hence any element of H_n is uniquely determined by its restrictions to those sets. Each $f_m^{(1)}$, $m \geq 1$ coincides with the identity map on $X \setminus W_1$. It follows that every map $h \in H$ also coincides with the identity map on $X \setminus W_1$. By the above the restrictions of $f_m^{(1)}$ to $W_1 \setminus W_2, W_2 \setminus W_3, \dots, W_{n-1} \setminus W_n$ coincide with elements of respectively $\widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_{n-1}$. Since each $\widetilde{F}_k$ is a group, it follows that the restrictions of any map $h \in H$ to $W_1 \setminus W_2, W_2 \setminus W_3, \dots, W_{n-1} \setminus W_n$ also coincide with elements of respectively $\widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_{n-1}$. Finally, each of the maps $f_1^{(1)}, f_2^{(1)}, \dots, f_n^{(1)}$ coincides on W_n with an element of the set F_n. Since F_n is a group, it follows that the restriction of any map $h \in H_n$ to W_n coincides with an element of F_n. As the groups $\widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_{n-1}$ and F_n are clearly finite, we conclude that the group H_n is finite as well. $\square$

Proposition 8.2. *Suppose X is a Cantor set and $\mathcal{G} \subset \mathrm{Homeo}(X)$ is an ample group that acts minimally on X. Then there are uncountably many infinite, nowhere dense closed sets $Y \subset X$ such that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y.*

Proof. Let $\rho : X \times X \to \mathbb{R}$ be any distance function on the Cantor set X compatible with its topology. For any $x \in X$ and $r > 0$ denote by $B(x, r)$ the ball of radius r centered at the point x: $B(x, r) = \{y \in X \mid \rho(y, x) < r\}$. The ball $B(x, r)$ is an open neighborhood of x of diameter at most $2r$.

We are going to construct a nested sequence $U_0 \supset U_1 \supset U_2 \supset \dots$ of nonempty clopen subsets of X and two sequences, $g_1, g_2, g_3, \dots$ and $h_1, h_2, h_3, \dots$, of elements of the group $\mathcal{G}$. The elements of $\mathcal{G}$ will be chosen so that for any $n \geq 1$ the sets U_n, $g_n(U_n)$ and $h_n(U_n)$ are disjoint subsets of U_{n-1}. Assuming this, generalized 2-cycles $f_n^{(1)} = \delta_{U_n; g_n}$ and $f_n^{(2)} = \delta_{U_n; h_n}$ are defined. Since $g_n, h_n \in \mathcal{G}$, the maps $f_n^{(1)}$ and $f_n^{(2)}$ belong to the group $\mathsf{F}(\mathcal{G}) = \mathcal{G}$. We also let $f_n^{(0)} = \mathrm{id}_X$. The clopen sets $U_0, U_1, U_2, \dots$ will be chosen so that for any string $\xi_1 \xi_2 \dots \xi_n$ of 0s, 1s and 2s, the set $f_1^{(\xi_1)} f_2^{(\xi_2)} \dots f_n^{(\xi_n)}(U_n)$ is of diameter at most 2^{-n}.

The construction is done inductively. First we let $U_0 = X$. Now assume that for some $n \geq 1$ the set U_{n-1} is already chosen and so are the maps g_k and h_k, $1 \leq k \leq n-1$. Since X is a Cantor set, the nonempty clopen set U_{n-1} is infinite. Take any three distinct points $y, y', y'' \in U_{n-1}$. Let ϵ be the least of the distances $\rho(y, y')$, $\rho(y, y'')$ and $\rho(y', y'')$. Then sets $V = B(y, \epsilon/2) \cap U_{n-1}$, $V' = B(y', \epsilon/2) \cap U_{n-1}$ and $V'' = B(y'', \epsilon/2) \cap U_{n-1}$ are open neighborhoods of respectively y, y' and y''. The triangle inequality for the distance function ρ implies that V, V' and V'' are disjoint subsets of U_{n-1}. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(y)$ is dense in X. In particular, this orbit has a point in V' and in V''. Therefore we can choose $g_n, h_n \in \mathcal{G}$ such that $g_n(y) \in V'$ and $h_n(y) \in V''$. Since g_n and h_n are continuous maps, the set $W = V \cap g_n^{-1}(V') \cap h_n^{-1}(V'')$ is an open neighborhood of the point y. By construction, $W \subset V$, $g_n(W) \subset V'$ and $h_n(W) \subset V''$, which implies that W, $g_n(W)$ and $h_n(W)$ are disjoint subsets of U_{n-1}. Now that the maps g_k and h_k are chosen for any k, $1 \leq k \leq n$, we can define the maps $f_k^{(i)}$, $i \in \{0, 1, 2\}$, $1 \leq k \leq n$ as described above. For any string $w = \xi_1 \xi_2 \dots \xi_n$ of 0s, 1s and 2s that has exactly n characters, we let $f^{(w)} = f_1^{(\xi_1)} f_2^{(\xi_2)} \dots f_n^{(\xi_n)}$ and $y_w = f^{(w)}(y)$. Further, let $\widetilde{W}$ be the intersection of W and all sets of the form $\big(f^{(w)}\big)^{-1}\big(B(y_w, 2^{-n-1})\big)$. Since each $f^{(w)}$ is a continuous map, it follows that $\widetilde{W}$ is an open neighborhood of the point y. As X is a Cantor set, the set $\widetilde{W}$ contains a clopen neighborhood of y. We choose the latter as U_n. The clopen set U_n is not empty since $y \in U_n$. Since $U_n \subset \widetilde{W} \subset W$, the sets U_n, $g_n(U_n)$ and $h_n(U_n)$ are disjoint subsets of

U_{n-1}. By construction, $f^{(w)}(U_n) \subset B(y_w, 2^{-n-1})$ for any string $w = \xi_1\xi_2\ldots\xi_n$ of 0s, 1s and 2s, which implies that the diameter of the set $f^{(w)}(U_n)$ is at most 2^{-n}. The inductive step of the construction is complete.

Let $\Xi = \{0,1,2\}^{\mathbb{N}}$. We regard any element $\xi \in \Xi$ as an infinite string $\xi = \xi_1\xi_2\xi_3\ldots$, where each $\xi_i \in \{0,1,2\}$. Note that for any $n \geq 1$ the sets $f_n^{(0)}(U_n) = U_n$, $f_n^{(1)}(U_n) = g_n(U_n)$ and $f_n^{(2)}(U_n) = h_n(U_n)$ are subsets of U_{n-1}. Therefore for any infinite string $\xi = \xi_1\xi_2\xi_3\ldots$ of 0s, 1s and 2s we have a nested sequence of clopen sets $U_0 \supset f_1^{(\xi_1)}(U_1) \supset f_1^{(\xi_1)}f_2^{(\xi_2)}(U_2) \supset \ldots$. By construction, the diameters of those sets tend to 0. Hence their intersection consists of a single point, which we denote $\beta(\xi)$. Now we have a map $\beta : \Xi \to X$. Let $x_0 = \beta(000\ldots)$. The point x_0 is in the intersection of sets $U_0, U_1, U_2, \ldots$.

Consider the set $\Omega = \{1,2\}^{\mathbb{N}}$, which is a subset of Ξ. For any infinite string $\omega = \omega_1\omega_2\omega_3\ldots \in \Omega$, let H_ω denote a subgroup of $\mathcal{G}$ generated by elements $f_n^{(\omega_n)}$, $n \geq 1$. Further, let $Y_\omega = \overline{\mathrm{Orb}_{H_\omega}(x_0)}$. Then Y_ω is a closed subset of X invariant under the action of the group H_ω. Since for any $n \geq 1$ the sets U_n, $g_n(U_n)$ and $h_n(U_n)$ are disjoint subsets of U_{n-1}, it follows that the union of U_n and $f_n^{(\omega_n)}(U_n)$ is a proper subset of U_{n-1}. Now Lemma 8.1 implies that Y_ω is a Cantor set which is nowhere dense in X, and the group H_ω acts minimally when restricted to Y_ω. Since $H_\omega \subset \mathrm{St}_{\mathcal{G}}(Y_\omega)$, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y_\omega)$ also acts minimally when restricted to Y_ω.

As the set Ω is uncountable, it remains to demonstrate that $Y_\omega \neq Y_{\omega'}$ whenever ω and ω' are different elements of Ω. Just as in the proof of Lemma 8.1 we showed that the map α is one-to-one, we can show that the map β is one-to-one. Just as in the proof of Lemma 8.1 we showed that the map α is onto, we can show for any $\omega = \omega_1\omega_2\omega_3\ldots \in \Omega$ that the set Y_ω consists of all points of the form $\beta(\xi_1\xi_2\xi_3\ldots)$, where $\xi_n \in \{0, \omega_n\}$ for each $n \geq 1$. Since the map β is one-to-one, it follows that $Y_\omega \cap \beta(\Omega) = \{\beta(\omega)\}$. As a consequence, $Y_\omega \neq Y_{\omega'}$ whenever $\omega \neq \omega'$. ☐

9 Property E

The majority of our results on maximal subgroups of ample groups obtained in Section 7 require that the ample group has Property E (to be precise, these are Theorems 7.18, 7.19, 7.20 and 7.21). Hence they have no value until we provide examples of ample groups with this property. We should note that Property E is complex and its verification is not going to be easy. We begin with establishing a weakened version of Property E that involves the generalized symmetric groups.

Proposition 9.1. *Suppose a group $G \subset \mathrm{Homeo}(X)$ acts minimally on X. Then for any clopen sets U_1 and U_2 that intersect, the generalized symmetric group over the local group $\mathsf{F}_{U_1\cup U_2}(G)$ is generated by the union of generalized symmetric groups over $\mathsf{F}_{U_1}(G)$ and $\mathsf{F}_{U_2}(G)$: $\mathsf{S}(\mathsf{F}_{U_1\cup U_2}(G)) = \langle \mathsf{S}(\mathsf{F}_{U_1}(G)) \cup \mathsf{S}(\mathsf{F}_{U_2}(G)) \rangle$.*

Proof. For any clopen set $W \subset X$ let D_W denote the set of all generalized 2-cycles in the local group $\mathsf{F}_W(G)$. Since the group $\mathsf{F}_W(G)$ is ample, it follows from Lemma 6.2 that the generalized symmetric group $\mathsf{S}(\mathsf{F}_W(G))$ is generated by D_W. As a consequence, for any clopen sets $U_1, U_2 \subset X$ the group $\mathsf{S}(\mathsf{F}_{U_1 \cup U_2}(G))$ is generated by $D_{U_1 \cup U_2}$ while the group $\langle \mathsf{S}(\mathsf{F}_{U_1}(G)) \cup \mathsf{S}(\mathsf{F}_{U_2}(G)) \rangle$ is generated by $D_{U_1} \cup D_{U_2}$. Note that the sets D_{U_1} and D_{U_2} are contained in $D_{U_1 \cup U_2}$ since the local groups $\mathsf{F}_{U_1}(G)$ and $\mathsf{F}_{U_2}(G)$ are contained in $\mathsf{F}_{U_1 \cup U_2}(G)$. To prove the proposition, it is enough to show that, assuming the intersection $U_1 \cap U_2$ is nonempty, every element of $D_{U_1 \cup U_2}$ belongs to the group $\langle D_{U_1} \cup D_{U_2} \rangle$.

Take an arbitrary element $f \in D_{U_1 \cup U_2}$. We have $f = \delta_{U;f'}$ for some clopen set $U \subset X$ and homeomorphism $f' : X \to X$. Observe that the generalized 2-cycle $\delta_{U;f}$ is defined and $f = \delta_{U;f}$. Since the clopen set U is contained in the support of $\delta_{U;f}$ and $f \in \mathsf{F}_{U_1 \cup U_2}(G)$, we obtain that $U \subset U_1 \cup U_2$. The set $U_1 \cup U_2$ is the disjoint union of 3 clopen sets $P_1 = U_1 \setminus U_2$, $P_2 = U_2 \setminus U_1$ and $P_3 = U_1 \cap U_2$. Since $f(U_1 \cup U_2) = U_1 \cup U_2$, it is also the disjoint union of 3 clopen sets $f^{-1}(P_1)$, $f^{-1}(P_2)$ and $f^{-1}(P_3)$. For any $i, j \in \{1, 2, 3\}$ let $Q_{ij} = P_i \cap f^{-1}(P_j)$. Then $U_1 \cup U_2$ is the disjoint union of 9 clopen sets Q_{ij}, $i, j \in \{1, 2, 3\}$.

First we consider the principal case when $U \subset Q_{12}$. In this case, $U \subset U_1 \setminus U_2$ and $f(U) \subset U_2 \setminus U_1$. Given any point $x \in U$, we are going to construct a clopen neighborhood V_x of x such that $\delta_{W;f} \in \langle D_{U_1} \cup D_{U_2} \rangle$ for every clopen set $W \subset V_x$. Since the group G acts minimally on X, the orbit $\mathrm{Orb}_G(x)$ is dense in X. Hence it has a point in the nonempty clopen set $U_1 \cap U_2$, that is, $g_x(x) \in U_1 \cap U_2$ for some $g_x \in G$. Let $V_x = U \cap g_x^{-1}(U_1 \cap U_2)$. Then V_x is a clopen neighborhood of x. Moreover, $g_x(V_x) \subset U_1 \cap U_2$. For any clopen set $W \subset V_x$, the sets W, $g_x(W)$ and $f(W)$ are disjoint since $W \subset U_1 \setminus U_2$, $g_x(W) \subset U_1 \cap U_2$ and $f(W) \subset U_2 \setminus U_1$. Therefore the generalized permutation $h_\pi = \mu[W; \mathrm{id}_X, g_x, f; \pi]$ is defined for any permutation $\pi \in \mathsf{S}_3$. Since $(1\,3) = (1\,2)(2\,3)(1\,2)$, it follows from Lemma 3.4 that $h_{(1\,3)} = h_{(1\,2)} h_{(2\,3)} h_{(1\,2)}$. Observe that $h_{(1\,3)} = \delta_{W;f}$, $h_{(1\,2)} = \delta_{W;g_x}$ and $h_{(2\,3)} = \delta_{g_x(W); f g_x^{-1}}$. Then $\mathrm{supp}(h_{(1\,2)}) = W \cup g_x(W) \subset U_1$ and $\mathrm{supp}(h_{(2\,3)}) = g_x(W) \cup f(W) \subset U_2$, which means that $h_{(1\,2)} \in \mathsf{F}_{U_1}(G)$ and $h_{(2\,3)} \in \mathsf{F}_{U_2}(G)$. Hence $h_{(1\,2)} \in D_{U_1}$ and $h_{(2\,3)} \in D_{U_2}$. Consequently, the map $\delta_{W;f} = h_{(1\,2)} h_{(2\,3)} h_{(1\,2)}$ belongs to the group $\langle D_{U_1} \cup D_{U_2} \rangle$.

The clopen set U is covered by its clopen subsets V_x, $x \in U$ constructed above. Since U is compact, there are finitely many points $x_1, x_2, \dots, x_k \in U$ such that the sets V_{x_i}, $1 \le i \le k$ form a subcover. Now let $W_1 = V_{x_1}$ and $W_i = V_{x_i} \setminus (V_{x_1} \cup \dots \cup V_{x_{i-1}})$ for $i = 2, 3, \dots, k$. Then $W_1, W_2, \dots, W_k$ are clopen sets that form a partition of U. By Lemma 6.3, $\delta_{U;f} = \delta_{W_1;f} \, \delta_{W_2;f} \dots \delta_{W_k;f}$. Since $W_i \subset V_{x_i}$ for $1 \le i \le k$, it follows from the above that $\delta_{W_i;f} \in \langle D_{U_1} \cup D_{U_2} \rangle$ for $1 \le i \le k$. Then $f = \delta_{U;f} \in \langle D_{U_1} \cup D_{U_2} \rangle$ as well.

Next we consider the case when $U \subset Q_{21}$. In this case, $U \subset U_2 \setminus U_1$ and $f(U) \subset U_1 \setminus U_2$. Since $f^{-1} = f$, it follows that $f(U) \subset Q_{12}$. Also, it follows that the generalized 2-cycle $\delta_{f(U);f}$ is defined and $\delta_{f(U);f} = \delta_{U;f} = f$. Hence this case is reduced to the previous one. By the above, $f \in \langle D_{U_1} \cup D_{U_2} \rangle$.

Now we consider the general case. The clopen set U is the disjoint union of 9 clopen sets $U \cap Q_{ij}$, $i, j \in \{1, 2, 3\}$. By Lemma 6.3, the generalized 2-cycle $f = \delta_{U;f}$ is the product of 9 commuting maps $f_{ij} = \delta_{U \cap Q_{ij};f}$, $i, j \in \{1, 2, 3\}$. All 9 maps belong to $D_{U_1 \cup U_2}$. The maps f_{12} and f_{21} are covered in the two cases considered before. By the above both maps belong to the group $\langle D_{U_1} \cup D_{U_2} \rangle$. As for the other 7 maps, note that $Q_{ij} \subset P_i$ and $f(Q_{ij}) \subset P_j$ for all $i, j \in \{1, 2, 3\}$. Therefore the support of f_{ij}, which is $(U \cap Q_{ij}) \cup f(U \cap Q_{ij})$, is contained in $P_i \cup P_j$. As a consequence, the supports of f_{11}, f_{13}, f_{31} and f_{33} are contained in U_1 while the supports of f_{22}, f_{23} and f_{32} are contained in U_2. Hence $f_{11}, f_{13}, f_{31}, f_{33} \in D_{U_1}$ and $f_{22}, f_{23}, f_{32} \in D_{U_2}$. We conclude that all 9 maps f_{ij}, $i, j \in \{1, 2, 3\}$ belong to the group

$\langle D_{U_1} \cup D_{U_2} \rangle$. Then $f \in \langle D_{U_1} \cup D_{U_2} \rangle$ as well. $\square$

If X is a finite set with the discrete topology, then all subsets of X are clopen. Hence the group $\mathrm{Homeo}(X)$ coincides with the symmetric group S_X of all permutations on X. Any permutation on X is also a generalized permutation (in the sense of Definition 3.2). Therefore for any group $G \subset \mathsf{S}_X$ the ample group $\mathsf{F}(G)$ coincides with the generalized symmetric group $\mathsf{S}(G)$. In this case the conclusion of Proposition 9.1 means that the group G has Property E. The hypothesis of Proposition 9.1 is that G acts minimally on X. For a finite X, this means transitivity, that is, the entire set X forms a single orbit. Then every permutation on X is pointwise (and hence piecewise) an element of G, which implies that $\mathsf{F}(G) = \mathsf{S}_X$. Hence Theorems 7.19, 7.20 and 7.21 apply to the symmetric group S_X (Theorem 7.18 is vacuous if X is finite). One of the hypotheses of Theorem 7.21 is that the subgroup H of the ample group $\mathcal{G}$ contains a local group $\mathcal{G}_U$ for some clopen set U containing more than one point. In the case $\mathcal{G} = \mathsf{S}_X$, this is equivalent to the condition that H contains a transposition.

Another case when $\mathsf{F}(G) = \mathsf{S}(G)$ so that Proposition 9.1 yields Property E is when the ample group $\mathsf{F}(G)$ is simple. Indeed, $\mathsf{S}(G)$ is always a normal subgroup of $\mathsf{F}(G)$ (due to Lemma 3.7). One example of such a group is Thompson's group V.

In general, Proposition 9.1 does not yield Property E, but it helps in deriving this property from simpler properties introduced in Section 5.

Lemma 9.2. *For any ample group, Properties M and PE imply Property E.*

Proof. Suppose $\mathcal{G} \subset \mathrm{Homeo}(X)$ is an ample group with Properties M and PE. For any clopen sets $U_1, U_2 \subset X$ the local groups $\mathcal{G}_{U_1}$ and $\mathcal{G}_{U_2}$ are contained in the local group $\mathcal{G}_{U_1 \cup U_2}$. Assuming the intersection $U_1 \cap U_2$ is nonempty, we need to show that, conversely, every map $f \in \mathcal{G}_{U_1 \cup U_2}$ belongs to the group $\langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$. Since U_1 is a clopen set, $f \in \mathcal{G}$ and $\mathcal{G}$ has Property PE, there exists $g \in \mathcal{G}$ such that $g(U_{\mathrm{out}}) = U_{\mathrm{in}}$, where $U_{\mathrm{out}} = \{x \in U_1 \mid f(x) \notin U_1\}$ and $U_{\mathrm{in}} = \{x \notin U_1 \mid f(x) \in U_1\}$. The sets U_{out} and U_{in} are clopen as $U_{\mathrm{out}} = U_1 \cap f^{-1}(X \setminus U_1)$ and $U_{\mathrm{in}} = (X \setminus U_1) \cap f^{-1}(U_1)$. Hence the generalized 2-cycle $h = \delta_{U_{\mathrm{out}};g}$ is defined. Since h is piecewise an element of the group $\mathcal{G}$, we obtain that $h \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. The support $\mathrm{supp}(h) = U_{\mathrm{out}} \cup U_{\mathrm{in}}$ is clearly contained in the support of f, which implies that $h \in \mathcal{G}_{U_1 \cup U_2}$. Consequently, $h \in \mathsf{S}(\mathcal{G}_{U_1 \cup U_2})$. Since the group $\mathcal{G}$ has Property M and the intersection $U_1 \cap U_2$ is nonempty, Proposition 9.1 implies that the generalized symmetric group $\mathsf{S}(\mathcal{G}_{U_1 \cup U_2})$ is generated by its subgroups $\mathsf{S}(\mathcal{G}_{U_1})$ and $\mathsf{S}(\mathcal{G}_{U_2})$. Since $\mathsf{S}(\mathcal{G}_{U_1}) \subset \mathcal{G}_{U_1}$ and $\mathsf{S}(\mathcal{G}_{U_2}) \subset \mathcal{G}_{U_2}$, we conclude that $h \in \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$.

The set U_1 is the disjoint union of two sets U_{out} and $U_1 \setminus U_{\mathrm{out}}$ while the set $X \setminus U_1$ is the disjoint union of two sets U_{in} and $(X \setminus U_1) \setminus U_{\mathrm{in}}$. The generalized 2-cycle h maps U_{out} and U_{in} onto each other while fixing all points of $U_1 \setminus U_{\mathrm{out}}$ and $(X \setminus U_1) \setminus U_{\mathrm{in}}$. By definition of the sets U_{out} and U_{in}, we have $f(U_{\mathrm{out}}) \subset X \setminus U_1$, $f(U_1 \setminus U_{\mathrm{out}}) \subset U_1$, $f(U_{\mathrm{in}}) \subset U_1$ and $f((X \setminus U_1) \setminus U_{\mathrm{in}}) \subset X \setminus U_1$. As a consequence, $(fh)(U_1) \subset U_1$ and $(fh)(X \setminus U_1) \subset X \setminus U_1$. Since $fh \in \mathcal{G}_{U_1 \cup U_2}$, it follows that $(fh)(U_1) = U_1$, $(fh)(U_2 \setminus U_1) = U_2 \setminus U_1$ and $(fh)(X \setminus (U_1 \cup U_2)) = X \setminus (U_1 \cup U_2)$. Hence the map fh belongs to the individual stabilizer of the partition $X = U_1 \sqcup (U_2 \setminus U_1) \sqcup (X \setminus (U_1 \cup U_2))$. By Lemma 7.14, the stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2 \setminus U_1, X \setminus (U_1 \cup U_2))$ is the internal direct product of three local groups $\mathcal{G}_{U_1}$, $\mathcal{G}_{U_2 \setminus U_1}$ and $\mathcal{G}_{X \setminus (U_1 \cup U_2)}$. Therefore $fh = g_1 g_2 g_3$ for some $g_1 \in \mathcal{G}_{U_1}$, $g_2 \in \mathcal{G}_{U_2 \setminus U_1}$ and $g_3 \in \mathcal{G}_{X \setminus (U_1 \cup U_2)}$.

Note that $f, h, g_1, g_2 \in \mathcal{G}_{U_1 \cup U_2}$. Then $g_3 = (g_1 g_2)^{-1} f h \in \mathcal{G}_{U_1 \cup U_2}$ as well. Hence the support of g_3 is empty so that $g_3 = \mathrm{id}_X$. We obtain that $f = g_1 g_2 h^{-1}$. Since $g_1 \in \mathcal{G}_{U_1}$, $g_2 \in \mathcal{G}_{U_2 \setminus U_1} \subset \mathcal{G}_{U_2}$ and $h \in \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$, we conclude that $f \in \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$. ☐

Proposition 9.3. *For any ample group, Properties M and NC imply Property E.*

Proof. By Lemma 5.5, Property NC is equivalent to Property UR. For an ample group, Property UR implies Property PE due to Lemma 5.12. By Lemma 9.2, Properties M and PE imply Property E for any ample group. ☐

Property NC is much simpler than Property E and it is not hard to find groups of homeomorphisms with this property. For example, we can take a cyclic group generated by one minimal homeomorphism of the Cantor set (see Proposition 9.9 below).

Definition 9.4. A homeomorphism $f : X \to X$ is called **minimal** if there is no closed set $Y \subset X$ different from the empty set and X that is invariant under f: $f(Y) \subset Y$.

Lemma 9.5. *Any homeomorphism $f : X \to X$ is minimal if and only if the cyclic group $\langle f \rangle$ generated by f acts minimally on X.*

Proof. Assume that the action of the group $\langle f \rangle$ on X is not minimal. Then there exists a closed set $Y \subset X$ different from the empty set and X such that $g(Y) = Y$ for all $g \in \langle f \rangle$. In particular, $f(Y) = Y$ so that f is not a minimal homeomorphism.

Conversely, assume that the homeomorphism f is not minimal, that is, $f(Y) \subset Y$ for some closed set Y different from the empty set and X. Then $f^{k+1}(Y) \subset f^k(Y)$ for all $k \geq 1$. Hence $Y, f(Y), f^2(Y), \dots$ is a nested sequence of closed sets. Let Y_∞ denote the intersection of all these sets. Then Y_∞ is a closed set satisfying $f(Y_\infty) = Y_\infty$. It follows that $f^k(Y_\infty) = Y_\infty$ for all $k \in \mathbb{Z}$. Since X is compact, the set Y_∞ is nonempty. Besides, $Y_\infty \neq X$ as $Y_\infty \subset Y$. Thus the group $\langle f \rangle$ does not act minimally on X. ☐

In view of Lemma 5.10, it follows from Lemma 9.5 that minimal homeomorphisms of X might exist only if the topological space X is finite or a Cantor set. We already discussed the case when X is finite earlier in this section. In what follows, we assume that X is a Cantor set.

There are many known examples of minimal homeomorphisms of Cantor sets (they are referred to as *Cantor minimal systems*). Suppose $f : X \to X$ is such a homeomorphism. Let us explore properties of the cyclic group $\langle f \rangle$ and its full amplification $\mathsf{F}(\langle f \rangle)$.

Lemma 9.6. *For any homeomorphism f of a Cantor set X there exists a Borel probability measure on X invariant under the action of the cyclic group $\langle f \rangle$.*

Proof. We are going to use the fact (due to von Neumann) that the group $\mathbb{Z}$ is *amenable*. This means that $\mathbb{Z}$ admits an *invariant mean*, which is a function $\mathfrak{m} : \mathcal{P}(\mathbb{Z}) \to [0, \infty)$ on the set $\mathcal{P}(\mathbb{Z})$ of all subsets of $\mathbb{Z}$ that is translation invariant, finitely additive, and satisfies $\mathfrak{m}(\mathbb{Z}) = 1$. Translation invariance means that $\mathfrak{m}(S+n) = \mathfrak{m}(S)$ for all sets $S \subset \mathbb{Z}$ and integers n, where $S + n = \{i + n \mid i \in S\}$. Finite additivity means that $\mathfrak{m}(S) = \mathfrak{m}(S_1) + \mathfrak{m}(S_2) + \dots + \mathfrak{m}(S_k)$ whenever $S = S_1 \sqcup S_2 \sqcup \dots \sqcup S_k$. Now choose any point $x \in X$ and define a map $\phi : \mathbb{Z} \to X$

by $\phi(i) = f^i(x)$ for all $i \in \mathbb{Z}$. Then for any set $E \subset X$ let $\nu_0(E) = \mathfrak{m}(\phi^{-1}(E))$. This defines a function $\nu_0 : \mathcal{P}(X) \to [0,\infty)$ on the set of all subsets of X. Clearly, $\nu_0(X) = \mathfrak{m}(\mathbb{Z}) = 1$ and $\nu_0(\emptyset) = \mathfrak{m}(\emptyset) = 0$. Whenever a set $E \subset X$ is the disjoint union of sets $E_1, E_2, \dots, E_k$, its preimage $\phi^{-1}(E)$ is the disjoint union of $\phi^{-1}(E_1), \phi^{-1}(E_2), \dots, \phi^{-1}(E_k)$. It follows that the function ν_0 is finitely additive. Further, $f^n(\phi(i)) = \phi(i+n)$ for all $i, n \in \mathbb{Z}$. It follows that $\phi^{-1}(f^n(E)) = \phi^{-1}(E) + n$ for all sets $E \subset X$ and integers n. Hence $\nu_0(f^n(E)) = \nu_0(E)$.

Let $\mathcal{CO}(X)$ denote the algebra of clopen subsets of X. The restriction of the function ν_0 to $\mathcal{CO}(X)$ is not only finitely additive but also σ-additive, which means that $\nu_0(U) = \nu_0(U_1)+\nu_0(U_2)+\nu_0(U_3)+\dots$ whenever $U = U_1 \sqcup U_2 \sqcup U_3 \sqcup \dots$. Indeed, since any clopen set U is compact, it can be the disjoint union of clopen sets $U_1, U_2, U_3, \dots$ only if at most finitely many of the latter are nonempty. Recall that the σ-algebra $\mathcal{B}(X)$ of the Borel subsets of the Cantor set X is generated by the algebra $\mathcal{CO}(X)$. It follows that the restriction $\nu_0|_{\mathcal{CO}(X)}$ is uniquely extended to a σ-additive function $\nu : \mathcal{B}(X) \to [0,\infty)$. Since $\nu(X) = \nu_0(X) = 1$, the function ν is a Borel probability measure on X. Note that for any integer n, the homeomorphism f^n maps clopen sets to clopen sets, Borel sets to Borel sets, and any disjoint sets to disjoint sets. As a consequence, the function $\mathcal{B}(X) \ni E \mapsto \nu(f^n(E))$ is also a Borel probability measure on X and, moreover, this measure coincides with ν_0 on $\mathcal{CO}(X)$. Uniqueness of the extension ν implies that $\nu(f^n(E)) = \nu(E)$ for all Borel sets $E \subset X$. Thus the measure ν is invariant under the action of the cyclic group $\langle f \rangle$ on X. $\square$

Lemma 9.7. *Suppose f is a minimal homeomorphism of a Cantor set X. Let p be a prime number. Then either f^p is also a minimal homeomorphism of X or else there exists a clopen set $U \subset X$ such that $X = U \sqcup f(U) \sqcup \dots \sqcup f^{p-1}(U)$. For any such clopen set U, the restriction of the map f^p to U is a minimal homeomorphism of U.*

Proof. Assume that f^p is not a minimal homeomorphism of X. Then there exists a closed set Y different from the empty set and X such that $f^p(Y) \subset Y$. It follows from Lemma 9.5 that Y can be chosen so that $f^p(Y) = Y$. Then $f^{a+kp}(Y) = f^a(Y)$ for all $a, k \in \mathbb{Z}$. Hence, given a coset $\alpha \in \mathbb{Z}/p\mathbb{Z}$, the image $f^a(Y)$ is the same for all $a \in \alpha$. We denote that image by Y_α. Further, for any nonempty set $S \subset \mathbb{Z}/p\mathbb{Z}$ we denote by Y_S the intersection of sets Y_α, $\alpha \in S$. Each Y_S is a closed set different from X. Clearly, $Y_{S_1 \cup S_2} = Y_{S_1} \cap Y_{S_2}$ for any nonempty sets $S_1, S_2 \subset \mathbb{Z}/p\mathbb{Z}$. Note that the set $Y_{\{p\mathbb{Z}\}} = Y_{p\mathbb{Z}} = Y$ is nonempty. Let S_0 be a subset of $\mathbb{Z}/p\mathbb{Z}$ of largest possible cardinality such that $Y_{S_0} \neq \emptyset$. We are going to show that $X = Y_{S_0} \sqcup f(Y_{S_0}) \sqcup \dots \sqcup f^{p-1}(Y_{S_0})$.

For any integer i and coset $\alpha = \ell + p\mathbb{Z}$, let $\alpha + i = (\ell + i) + p\mathbb{Z} = \{a + i \mid a \in \alpha\}$. Then $f^i(Y_\alpha) = Y_{\alpha+i}$. Further, for any $i \in \mathbb{Z}$ and nonempty set $S \subset \mathbb{Z}/p\mathbb{Z}$ let $S+i = \{\alpha+i \mid \alpha \in S\}$. Then $f^i(Y_S) = Y_{S+i}$. Since $\mathbb{Z}/p\mathbb{Z} + 1 = \mathbb{Z}/p\mathbb{Z}$, we obtain that $f(Y_{\mathbb{Z}/p\mathbb{Z}}) = Y_{\mathbb{Z}/p\mathbb{Z}}$. Minimality of f implies that $Y_{\mathbb{Z}/p\mathbb{Z}} = \emptyset$. As a consequence, S_0 is a proper subset of $\mathbb{Z}/p\mathbb{Z}$. Since p is a prime number, it follows that $S_0 + i \neq S_0 + j$ for any integers i and j not congruent modulo p. Indeed, otherwise we would have $S_0 = S_0 + k(j-i)$ for all integers $k \geq 1$, which would imply that $S_0 = \mathbb{Z}/p\mathbb{Z}$. Since the sets $S_0 + i$ and $S_0 + j$ have the same number of elements as S_0, their union has strictly more elements. By the choice of S_0, we obtain that $Y_{(S_0+i)\cup(S_0+j)} = \emptyset$. Since $Y_{(S_0+i)\cup(S_0+j)} = Y_{S_0+i} \cap Y_{S_0+j}$, the sets Y_{S_0+i} and Y_{S_0+j} are disjoint. In particular, $Y_{S_0}, Y_{S_0+1}, \dots, Y_{S_0+p-1}$ are pairwise disjoint sets. Let $Y^+ = Y_{S_0} \cup Y_{S_0+1} \cup \dots \cup Y_{S_0+p-1}$. We have $f(Y_{S_0+i}) = Y_{S_0+i+1}$ for $i = 0, 1, \dots, p-2$ and $f(Y_{S_0+p-1}) = Y_{S_0+p} = Y_{S_0}$. Therefore

$f(Y^+) = Y^+$. Since Y^+ is a nonempty closed set, minimality of the homeomorphism f implies that $Y^+ = X$. Thus $X = Y_{S_0} \sqcup Y_{S_0+1} \sqcup \cdots \sqcup Y_{S_0+p-1} = Y_{S_0} \sqcup f(Y_{S_0}) \sqcup \cdots \sqcup f^{p-1}(Y_{S_0})$. Since the sets $Y_{S_0}, f(Y_{S_0}), \ldots, f^{p-1}(Y_{S_0})$ are closed, it follows that they are also open.

Now assume that $X = U \sqcup f(U) \sqcup \cdots \sqcup f^{p-1}(U)$ for some clopen set $U \subset X$. Since the map f is invertible, we obtain $X = f(X) = f(U) \sqcup f^2(U) \sqcup \cdots \sqcup f^p(U)$, which implies that $f^p(U) = U$. Note that U is different from the empty set and X. Therefore f^p is not a minimal homeomorphism of X. Further, $f^{-p}(U) = U$. Clearly, the restriction of f^{-p} to U is the inverse of the restriction of f^p to U. Hence $f^p|_U$ is a homeomorphism of U. Suppose Y is a closed subset of U such that $f^p(Y) \subset Y$. Then $Y^+ = Y \cup f(Y) \cup \cdots \cup f^{p-1}(Y)$ is a closed subset of X such that $f(Y^+) \subset Y^+$. Minimality of the homeomorphism f implies that Y^+ is empty or $Y^+ = X$. Since $Y = Y^+ \cap U$, it follows that the set Y is empty or $Y = U$. Thus $f^p|_U$ is a minimal homeomorphism of U. □

Lemma 9.8. *Suppose f is a minimal homeomorphism of a Cantor set X. For any integer $n \neq 0$ there exists a partition of X into nonempty clopen sets $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ such that the restriction of the map f^n to each U_i, $1 \leq i \leq k$ is a minimal homeomorphism of U_i.*

Proof. Since the inverse map f^{-1} generates the same cyclic group as f, it follows from Lemma 9.5 that f^{-1} is also a minimal homeomorphism of X. As $f^{-n} = (f^{-1})^n$ for all $n > 0$, this allows to reduce the general case to the case $n > 0$.

If f^n is a minimal homeomorphism of X (e.g., for $n = 1$), we can take the trivial partition $\{X\}$. If f^n is not a minimal homeomorphism of X and n is a prime number, it follows from Lemma 9.7 that $X = U \sqcup f(U) \sqcup \cdots \sqcup f^{p-1}(U)$ for some clopen set $U \subset X$. Moreover, for any such clopen set U the restriction of f^n to U is a minimal homeomorphism of U. Note that $X = f^i(X) = f^i(U) \sqcup f^{i+1}(U) \sqcup \cdots \sqcup f^{i+p-1}(U) = f^i(U) \sqcup f(f^i(U)) \sqcup \cdots \sqcup f^{p-1}(f^i(U))$ for $i = 0, 1, \ldots, p-1$. Therefore the restriction of the map f^n to each of the disjoint nonempty clopen sets $U, f(U), \ldots, f^{p-1}(U)$ is a minimal homeomorphism.

The proof for all $n > 0$ is by strong induction on n. The cases when $n = 1$ or n is prime have already been settled. Otherwise $n = n_1 n_2$ for some integers n_1 and n_2, $1 < n_1, n_2 < n$. By the inductive assumption, there exists a partition $\{U_1, U_2, \ldots, U_k\}$ of X into nonempty clopen sets such that the restriction of the map f^{n_1} to each U_i is a minimal homeomorphism of U_i. Again by the inductive assumption, for any i, $1 \leq i \leq k$ there exists a partition $U_i = U_{i1} \sqcup U_{i2} \sqcup \cdots \sqcup U_{i\ell_i}$ into nonempty clopen sets such that the restriction of the map $(f^{n_1})^{n_2} = f^n$ to each U_{ij}, $1 \leq j \leq \ell_i$ is a minimal homeomorphism of U_{ij}. It remains to notice that the clopen sets U_{ij}, $1 \leq i \leq k$, $1 \leq j \leq \ell_i$ form a partition of X. □

Proposition 9.9. *For any minimal homeomorphism f of a Cantor set X, the cyclic group $\langle f \rangle$ is infinite and has Properties FrA, M, IM, UR and NC.*

Proof. By Lemma 9.6, the action of the group $\langle f \rangle$ on X admits an invariant Borel probability measure, that is, $\langle f \rangle$ has Property IM. By Lemma 9.5, the action on X is minimal, that is, $\langle f \rangle$ has Property M. Then each orbit of $\langle f \rangle$ is dense in X (due to Lemma 5.9). Since the Cantor set X is infinite, each orbit is infinite as well. It follows that $f^n(x) \neq f^m(x)$ for all $x \in X$ whenever $n \neq m$. As a consequence, the group $\langle f \rangle$ is infinite and its action on X is free (Property FrA).

By Lemma 5.5, Properties NC and UR are equivalent. To establish Property UR for the group $\langle f\rangle$, we need to show for any point $x \in X$, any open neighborhood U of x, and any integer $n \neq 0$ that there exists an integer $k \geq 1$ such that $(f^n)^k(x) \in U$. By Lemma 9.8, there is a partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_\ell$ such that each U_i, $1 \leq i \leq \ell$ is a nonempty clopen set and the restriction of the map f^n to U_i is a minimal homeomorphism of U_i. Let V denote the element of the partition that contains the point x. Then $U \cap V$ is an open neighborhood of x. Clearly, $f^n(V) = V$. Hence the sequence $f^n(x), f^{2n}(x), f^{3n}(x), \dots$ is contained in V. Let S be the set of all points in this sequence. Since V is a clopen set, the closure $\overline{S}$ is contained in V as well. Note that $f^n(S) \subset S$. Since f^n is a homeomorphism, it follows that $f^n(\overline{S}) = \overline{f^n(S)} \subset \overline{S}$. As $\overline{S}$ is a closed subset of V, minimality of $f^n|_V$ implies that $\overline{S} = V$. Therefore the sequence $f^n(x), f^{2n}(x), f^{3n}(x), \dots$ is dense in V. In particular, it visits the nonempty open set $U \cap V$. Thus $(f^n)^k(x) = f^{kn}(x) \in U \cap V \subset U$ for some $k \geq 1$. ☐

Proposition 9.10. *For any minimal homeomorphism f of a Cantor set X, the ample group $\mathsf{F}(\langle f\rangle)$ has Properties C, NSP, M, IM and NC–. That is, $\mathsf{F}(\langle f\rangle)$ is a countable group that acts on X minimally and without singular points, admits an invariant Borel probability measure, and does not contract clopen sets.*

Proof. By Proposition 9.9, the cyclic group $\langle f\rangle$ is infinite and has Properties FrA, M and IM. Since $\langle f\rangle \subset \mathsf{F}(\langle f\rangle)$, the group $\mathsf{F}(\langle f\rangle)$ is also infinite. Since $\langle f\rangle$ is countable, it follows from Lemma 5.1 that $\mathsf{F}(\langle f\rangle)$ is also countable (Property C). Property FrA clearly implies Property NSP. Since $\langle f\rangle$ has Property NSP, it follows from Lemma 5.3 that $\mathsf{F}(\langle f\rangle)$ has this property as well. Since the group $\langle f\rangle$ acts minimally on X (Property M) and $\langle f\rangle \subset \mathsf{F}(\langle f\rangle)$, the group $\mathsf{F}(\langle f\rangle)$ also acts minimally on X. Further, $\langle f\rangle$ admits an invariant Borel probability measure ν on X (Property IM). By Lemma 5.7, the same measure ν is invariant under the action of $\mathsf{F}(\langle f\rangle)$ on X. Finally, Properties IM and M imply Property IM+ (due to Lemma 5.11), which in turn implies Property NC– (due to Lemma 5.8). ☐

Theorem 9.11. *For any minimal homeomorphism f of a Cantor set, the ample group $\mathsf{F}(\langle f\rangle)$ has Property E.*

At the moment, we cannot derive Theorem 9.11 from Proposition 9.3. By Proposition 9.9, the cyclic group $\langle f\rangle$ has Property NC. The ample group $\mathsf{F}(\langle f\rangle)$ has Property NC– due to Proposition 9.10, but it is not clear whether it has Property NC. In general, it is not clear whether Property NC survives when the group is amplified.

We are going to derive Theorem 9.11 directly from Proposition 9.1. To do this, we will use some advanced results obtained by Giordano, Putnam, Skau [GPS] and by Matui [Mat].

Theorem 9.12 ([GPS])**.** *For any minimal homeomorphism f of a Cantor set, there exists a unique homomorphism $I : \mathsf{F}(\langle f\rangle) \to \mathbb{Z}$ such that $I(f) = 1$.*

The homomorphism I is called the *index map*. Clearly, every element of finite order in the group $\mathsf{F}(\langle f\rangle)$ is contained in the kernel of I.

Theorem 9.13 ([Mat])**.** *Let f be a minimal homeomorphism of a Cantor set. Then every element of the kernel of the index map $I : \mathsf{F}(\langle f\rangle) \to \mathbb{Z}$ is a product of two elements of finite order in $\mathsf{F}(\langle f\rangle)$.*

Theorem 9.12 is the essential part of Propositions 5.4 and 5.5 from the paper [GPS]. Theorem 9.13 is the essential part of Lemma 4.1 from the paper [Mat]. In addition to these theorems, we need another result.

Lemma 9.14. *Let f be a minimal homeomorphism of a Cantor set X. For any nonempty clopen set $U \subset X$ the map f can be written as $f = f_U h_U$, where $h_U \in \mathsf{S}(\langle f \rangle)$, $f_U \in \mathsf{F}_U(\langle f \rangle)$ and, moreover, $\mathsf{F}_U(\langle f \rangle) = \mathsf{F}(\langle f_U \rangle)$.*

Proof. We are going to utilize the Kakutani-Rokhlin partition relative to the set U and map f. This construction was used previously in the proof of Lemma 5.12. Let us describe it briefly. We begin with the set $U_\pm$ of all points $x \in X$ such that each of two sequences $f(x), f^2(x), f^3(x), \dots$ and $x, f^{-1}(x), f^{-2}(x), \dots$ visits U at least once. For any $x \in U_\pm$ let $p(x)$ denote the least positive integer k such that $f^k(x) \in U$ and let $m(x)$ denote the least nonnegative integer k such that $f^{-k}(x) \in U$. Further, for any integers k and ℓ, $k \geq \ell \geq 1$, let $W_{k,\ell}$ denote the set of all points $x \in U_\pm$ such that $p(x) + m(x) = k$ and $m(x) + 1 = \ell$. It was shown in the proof of Lemma 5.12 that the sets $W_{k,\ell}$, $k \geq \ell \geq 1$ are clopen, the set $U_\pm$ is the disjoint union of all of them, and $U \cap U_\pm$ is the disjoint union of sets $W_{k,1}$, $k \geq 1$. Also, it was shown that $f(W_{k,\ell}) = W_{k,\ell+1}$ if $\ell < k$, and $f(W_{k,k}) \subset U$.

By Proposition 9.9, the cyclic group $\langle f \rangle$ has Property UR. Since $f \in \langle f \rangle$, it follows that $U \subset U_\pm$ and $f(U_\pm) \subset U_\pm$. As a consequence, the compact set U is the disjoint union of clopen sets $W_{k,1}$, $k \geq 1$. Therefore all but finitely many of the latter are empty. If $W_{k,1}$ is empty for some k, then the sets $W_{k,2}, W_{k,3}, \dots, W_{k,k}$ are empty as well. It follows that only finitely many sets of the form $W_{k,\ell}$ are nonempty. Hence their union $U_\pm$ is clopen. Since $f(U_\pm) \subset U_\pm$, minimality of f implies that $U_\pm = X$.

For any $k \geq 1$ the sets $W_{k,1}, W_{k,2}, \dots, W_{k,k}$ are disjoint. Since $f(W_{k,\ell}) = W_{k,\ell+1}$ if $\ell < k$, it follows that $W_{k,i} = f^{i-1}(W_{k,1})$ for $i = 1, 2, \dots, k$. Therefore the generalized permutation $h_{U,k} = \mu[W_{k,1}; \mathrm{id}_X, f, f^2, \dots, f^{k-1}; (1\,2 \dots k)]$ is defined. By definition, $h_{U,k}$ is piecewise an element of the group $\langle f \rangle$. Then $h_{U,k}$ belongs to the group $\mathsf{F}(\langle f \rangle)$ and hence also to $\mathsf{S}(\langle f \rangle)$. The maps $h_{U,1}, h_{U,2}, h_{U,3}, \dots$ have pairwise disjoint supports. Since only finitely many sets of the form $W_{k,\ell}$ are nonempty, there exists $k_0 \geq 1$ such that $W_{k,\ell} = \emptyset$ whenever $k > k_0$. Then $h_{U,k} = \mathrm{id}_X$ for all $k > k_0$. Now let $h_U = h_{U,1} h_{U,2} \dots h_{U,k_0}$. The map h_U belongs to $\mathsf{S}(\langle f \rangle)$.

Next we define a map $f_U : X \to X$. For any $x \in X$, let $f_U(x) = f^{p(x)}(x)$ if $x \in U$ and $f(x) = x$ otherwise. Notice that the restriction of the map f_U to the set U is the *first-return map* of f relative to U. Indeed, for any $x \in U$ the point $f_U(x)$ is the first element of the sequence $f(x), f^2(x), f^3(x), \dots$ that belongs to U. Let us show that $f = f_U h_U$. Any point $x \in X$ belongs to a unique set $W_{k,\ell}$. The map h_U coincides with $h_{U,k}$ on $W_{k,\ell}$. If $\ell < k$ then $h_U(x) = f(x)$, which belongs to $W_{k,\ell+1}$. The set $W_{k,\ell+1}$ is disjoint from U since $\ell + 1 \neq 1$. Hence $f_U(h_U(x)) = h_U(x) = f(x)$. In the case $\ell = k$, we have $h_U(x) = f^{1-k}(x)$, which is in $W_{k,1}$. The set $W_{k,1}$ is contained in U. Since $p(y) + m(y) = k$ and $m(y) + 1 = 1$ for all $y \in W_{k,1}$, it follows that $p(h_U(x)) = k$. Then $f_U(h_U(x)) = f^k(h_U(x)) = f^k(f^{1-k}(x)) = f(x)$.

Since $f = f_U h_U$, it follows that $f_U = f h_U^{-1}$. Therefore f_U is a homeomorphism and an element of the group $\mathsf{F}(\langle f \rangle)$. By construction, $\mathrm{supp}(f_U) \subset U$ so that $f_U \in \mathsf{F}_U(\langle f \rangle)$. Since the local group $\mathsf{F}_U(\langle f \rangle)$ is ample, it follows that $\mathsf{F}(\langle f_U \rangle) \subset \mathsf{F}_U(\langle f \rangle)$. It remains to show that $\mathsf{F}_U(\langle f \rangle) \subset \mathsf{F}(\langle f_U \rangle)$ or, equivalently, that any map $g \in \mathsf{F}_U(\langle f \rangle)$ is locally an element of the cyclic group $\langle f_U \rangle$. On the set $X \setminus U$, the map g coincides with id_X, which belongs to $\langle f_U \rangle$.

For any point $x \in U$ we have $g(x) \in U$ and, moreover, g coincides with some f^n in an open neighborhood of x. Therefore it is enough to prove the following claim: for any $x \in U$ and $n \in \mathbb{Z}$ such that $f^n(x) \in U$, there exists $s \in \mathbb{Z}$ such that the map f^n coincides with f_U^s in an open neighborhood of the point x.

In the case $n \geq 0$, the claim is proved by strong induction on n. For $n = 0$, we can take $s = 0$. For $n > 0$, we are going to apply the inductive assumption to the point $y = f_U(x)$. Since $x \in U$, we have $x \in W_{k,1}$ for some $k \geq 1$. The function p takes the constant value k on the clopen set $W_{k,1}$. Hence the map f_U coincides with f^k on $W_{k,1}$. In particular, $y = f^k(x)$. Then $f^n(x) = f^{n-k}(y)$. Note that $k = p(x) \leq n$ as $f^n(x) \in U$. Therefore $0 \leq n - k < n$. By the inductive assumption, for some integer s the map f^{n-k} coincides with f_U^s in an open neighborhood V of the point y. Then the map $f^n = f^{n-k} f^k$ coincides with $f_U^s f_U = f_U^{s+1}$ on the set $W_{k,1} \cap f_U^{-1}(V)$, which is an open neighborhood of x.

In the case $n < 0$, let $y = f^n(x)$. Then $y \in U$ and $x = f^{-n}(y)$. Since $-n > 0$, it follows from the above that for some $s \in \mathbb{Z}$ the map f^{-n} coincides with f_U^s in an open neighborhood V' of the point y. Then the inverse maps $(f^{-n})^{-1} = f^n$ and $(f_U^s)^{-1} = f_U^{-s}$ coincide on the set $f^{-n}(V')$, which is an open neighborhood of x. This completes the proof of the claim. $\square$

Let us collect all facts about the group $\mathsf{F}(\langle f \rangle)$ relevant to the proof of Theorem 9.11 into a single proposition.

Proposition 9.15. *Let f be a minimal homeomorphism of a Cantor set X. Then*

(i) the groups $\mathsf{S}(\langle f \rangle)$, $\mathsf{Fin}(\langle f \rangle)$ and $\mathsf{K}(\langle f \rangle)$ are the same;

(ii) the factor group $\mathsf{F}(\langle f \rangle)/\mathsf{S}(\langle f \rangle)$ is cyclic, generated by the coset of f;

(iii) the coset $f\mathsf{S}(\langle f \rangle)$ has an element in the local group $\mathsf{F}_U(\langle f \rangle)$ for any nonempty clopen set $U \subset X$;

(iv) $\mathsf{S}\big(\mathsf{F}_U(\langle f \rangle)\big) = \mathsf{S}(\langle f \rangle) \cap \mathsf{F}_U(\langle f \rangle)$ for any nonempty clopen set $U \subset X$.

Proof. By Proposition 9.9, the cyclic group $\langle f \rangle$ has Property FrA, which implies Property NSP. Then it follows from Proposition 5.4 that the group $\mathsf{S}(\langle f \rangle)$ generated by all generalized permutations in $\mathsf{F}(\langle f \rangle)$ coincides with the group $\mathsf{Fin}(\langle f \rangle)$ generated by all elements of finite order in $\mathsf{F}(\langle f \rangle)$. The group $\mathsf{K}(\langle f \rangle)$ is the intersection of the kernels of all nontrivial homomorphisms of the ample group $\mathsf{F}(\langle f \rangle)$ to $\mathbb{Z}$. It is easy to see that every element of finite order in $\mathsf{F}(\langle f \rangle)$ belongs to $\mathsf{K}(\langle f \rangle)$. Hence $\mathsf{Fin}(\langle f \rangle) \subset \mathsf{K}(\langle f \rangle)$. Since the index map $I : \mathsf{F}(\langle f \rangle) \to \mathbb{Z}$ provided by Theorem 9.12 is a nontrivial homomorphism, the group $\mathsf{K}(\langle f \rangle)$ is contained in its kernel $\ker(I)$. By Theorem 9.13, every element of $\ker(I)$ is a product of two elements of finite order in $\mathsf{F}(\langle f \rangle)$. Therefore $\ker(I) \subset \mathsf{Fin}(\langle f \rangle)$. We conclude that $\mathsf{Fin}(\langle f \rangle) = \mathsf{K}(\langle f \rangle) = \ker(I)$.

By Lemma 3.7, the group $\mathsf{S}(\langle f \rangle)$ is a normal subgroup of $\mathsf{F}(\langle f \rangle)$. Since $\mathsf{S}(\langle f \rangle)$ coincides with the kernel of the homomorphism I, the factor group $\mathsf{F}(\langle f \rangle)/\mathsf{S}(\langle f \rangle)$ is isomorphic to the image of I, which is $\mathbb{Z}$. Hence $\mathsf{F}(\langle f \rangle)/\mathsf{S}(\langle f \rangle)$ is cyclic. Moreover, it is generated by the coset $f\mathsf{S}(\langle f \rangle)$ since $I(f) = 1$.

Consider any nonempty clopen set $U \subset X$. By Lemma 9.14, we have $f = f_U h_U$, where $h_U \in \mathsf{S}(\langle f\rangle)$, $f_U \in \mathsf{F}_U(\langle f\rangle)$ and, moreover, $\mathsf{F}_U(\langle f\rangle) = \mathsf{F}(\langle f_U\rangle)$. Then $f_U = f h_U^{-1}$ so that $f_U \in f\mathsf{S}(\langle f\rangle)$. Since $\mathsf{S}(\langle f\rangle) = \ker(I)$, we obtain $I(f_U) = I(f) - I(h_U) = I(f) = 1$.

It remains to prove that $\mathsf{S}\big(\mathsf{F}_U(\langle f\rangle)\big) = \mathsf{S}(\langle f\rangle) \cap \mathsf{F}_U(\langle f\rangle)$. First we show that the local group $\mathsf{F}_U(\langle f\rangle)$ acts minimally when restricted to U. By Proposition 9.10, the group $\mathsf{F}(\langle f\rangle)$ acts minimally on X. It follows from Lemma 7.8 that the stabilizer $\mathrm{St}_{\mathsf{F}(\langle f\rangle)}(U)$ acts minimally on U. Since $g(U) = U$ if and only if $g(X \setminus U) = X \setminus U$ for any invertible map $g : X \to X$, the set stabilizer $\mathrm{St}_{\mathsf{F}(\langle f\rangle)}(U)$ coincides with the individual stabilizer $\mathrm{St}^{\bullet}_{\mathsf{F}(\langle f\rangle)}(U, X \setminus U)$ of the partition $X = U \sqcup X \setminus U$. By Lemma 7.14, $\mathrm{St}^{\bullet}_{\mathsf{F}(\langle f\rangle)}(U, X \setminus U)$ is the internal direct product of the local groups $\mathsf{F}_U(\langle f\rangle)$ and $\mathsf{F}_{X\setminus U}(\langle f\rangle)$. Since the action of $\mathsf{F}_{X\setminus U}(\langle f\rangle)$ on U is trivial, it follows that the action of $\mathsf{F}_U(\langle f\rangle)$ on U is minimal.

Note that the set U with the induced topology is a Cantor set. The group $\mathrm{Homeo}(U)$ of homeomorphisms of U is naturally isomorphic to the local group $\mathrm{Homeo}(X)_U$ consisting of all elements of $\mathrm{Homeo}(X)$ with supports in U. The isomorphism $\Phi : \mathrm{Homeo}(X)_U \to \mathrm{Homeo}(U)$ is given by $\Phi(g) = g|_U$ for all $g \in \mathrm{Homeo}(X)_U$. Any two maps in $\mathrm{Homeo}(X)_U$ coincide on a set $E \subset X$ if and only if their restrictions to U coincide on $E \cap U$. This fact implies that $\Phi(\mathsf{F}(H)) = \mathsf{F}(\Phi(H))$ for all groups $H \subset \mathrm{Homeo}(X)_U$. In particular, $\Phi\big(\mathsf{F}(\langle f_U\rangle)\big) = \mathsf{F}\big(\Phi(\langle f_U\rangle)\big) = \mathsf{F}(\langle f_U|_U\rangle)$. Further, a map $g \in \mathrm{Homeo}(X)_U$ is a generalized permutation on X if and only if $g|_U$ is a generalized permutation on U. This fact implies that $\Phi(\mathsf{S}(H)) = \mathsf{S}(\Phi(H))$ for all groups $H \subset \mathrm{Homeo}(X)_U$. In particular, $\Phi\big(\mathsf{S}(\langle f_U\rangle)\big) = \mathsf{S}\big(\Phi(\langle f_U\rangle)\big) = \mathsf{S}(\langle f_U|_U\rangle)$.

Next we show that $f_U|_U$ is a minimal homeomorphism of U. By the above the group $\mathsf{F}_U(\langle f\rangle) = \mathsf{F}(\langle f_U\rangle)$ acts minimally on U. Clearly, a set $E \subset U$ is invariant under the action of $\mathsf{F}(\langle f_U\rangle)$ if and only if it is invariant under the action of $\Phi\big(\mathsf{F}(\langle f_U\rangle)\big) = \mathsf{F}(\langle f_U|_U\rangle)$. Hence the group $\mathsf{F}(\langle f_U|_U\rangle)$ also acts minimally on U. By Lemma 2.8, the cyclic group $\langle f_U|_U\rangle$ has the same orbits as $\mathsf{F}(\langle f_U|_U\rangle)$. Then it follows from Lemma 5.9 that $\langle f_U|_U\rangle$ also acts minimally on U. Finally, Lemma 9.5 implies that $f_U|_U$ is a minimal homeomorphism of U.

Since the restriction $f_U|_U$ is a minimal homeomorphism of the Cantor set U, the above argument applies to $f_U|_U$. In particular, we obtain that there exists a unique homomorphism $I_U : \mathsf{F}(\langle f_U|_U\rangle) \to \mathbb{Z}$ such that $I_U(f_U|_U) = 1$. Also, we obtain that $\ker(I_U) = \mathsf{S}(\langle f_U|_U\rangle)$. It follows that the map $J_U = \Phi^{-1} I_U \Phi$ is a unique homomorphism of the group $\mathsf{F}(\langle f_U\rangle)$ to $\mathbb{Z}$ such that $J_U(f_U) = 1$. Besides, $\ker(J_U) = \mathsf{S}(\langle f_U\rangle)$. Since $I(f_U) = 1$, uniqueness of J_U implies that the index map I coincides with J_U on $\mathsf{F}(\langle f_U\rangle)$. As a consequence, $\ker(J_U) = \ker(I) \cap \mathsf{F}(\langle f_U\rangle)$. Since $\mathsf{F}(\langle f_U\rangle) = \mathsf{F}_U(\langle f\rangle)$ and hence $\mathsf{S}(\langle f_U\rangle) = \mathsf{S}\big(\mathsf{F}_U(\langle f\rangle)\big)$, we obtain that $\mathsf{S}\big(\mathsf{F}_U(\langle f\rangle)\big) = \ker(I) \cap \mathsf{F}_U(\langle f\rangle) = \mathsf{S}(\langle f\rangle) \cap \mathsf{F}_U(\langle f\rangle)$. □

Now we are ready to prove Theorem 9.11.

Proof of Theorem 9.11. Let f be a minimal homeomorphism of a Cantor set X. To establish Property E for the ample group $\mathsf{F}(\langle f\rangle)$, we have to show that for any clopen sets $U_1, U_2 \subset X$ the local group $\mathsf{F}_{U_1\cup U_2}(\langle f\rangle)$ is generated by its subgroups $\mathsf{F}_{U_1}(\langle f\rangle)$ and $\mathsf{F}_{U_2}(\langle f\rangle)$ provided that $U_1 \cap U_2 \neq \emptyset$. By Proposition 9.15, the factor group $\mathsf{F}(\langle f\rangle)/\mathsf{S}(\langle f\rangle)$ is cyclic, generated by the coset of f. Moreover, the coset $f\mathsf{S}(\langle f\rangle)$ contains an element $f_1 \in \mathsf{F}_{U_1}(\langle f\rangle)$. Then $f\mathsf{S}(\langle f\rangle) = f_1\mathsf{S}(\langle f\rangle)$. It follows that any map $g \in \mathsf{F}(\langle f\rangle)$ can be written as $g = f_1^k h$ for some

$k \in \mathbb{Z}$ and $h \in \mathsf{S}(\langle f\rangle)$. Assuming g belongs to the local group $\mathsf{F}_{U_1\cup U_2}(\langle f\rangle)$, we obtain that so does $h = f_1^{-k}g$. By Proposition 9.15, the intersection of the groups $\mathsf{S}(\langle f\rangle)$ and $\mathsf{F}_{U_1\cup U_2}(\langle f\rangle)$ coincides with the generalized symmetric group $\mathsf{S}\big(\mathsf{F}_{U_1\cup U_2}(\langle f\rangle)\big)$. Hence $h \in \mathsf{S}\big(\mathsf{F}_{U_1\cup U_2}(\langle f\rangle)\big)$. Since the group $\langle f\rangle$ acts minimally on X (due to Lemma 9.5), it follows from Proposition 9.1 that the group $\mathsf{S}\big(\mathsf{F}_{U_1\cup U_2}(\langle f\rangle)\big)$ is generated by its subgroups $\mathsf{S}\big(\mathsf{F}_{U_1}(\langle f\rangle)\big)$ and $\mathsf{S}\big(\mathsf{F}_{U_2}(\langle f\rangle)\big)$. Finally, $\mathsf{S}\big(\mathsf{F}_{U_1}(\langle f\rangle)\big) \subset \mathsf{F}_{U_1}(\langle f\rangle)$ and $\mathsf{S}\big(\mathsf{F}_{U_2}(\langle f\rangle)\big) \subset \mathsf{F}_{U_2}(\langle f\rangle)$. We conclude that h belongs to the group $\big\langle \mathsf{F}_{U_1}(\langle f\rangle) \cup \mathsf{F}_{U_2}(\langle f\rangle)\big\rangle$. Then $g = f_1^k h$ is in that group as well. Thus $\mathsf{F}_{U_1\cup U_2}(\langle f\rangle) = \big\langle \mathsf{F}_{U_1}(\langle f\rangle) \cup \mathsf{F}_{U_2}(\langle f\rangle)\big\rangle$. $\square$

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