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Planar structures and billiards in rational polygons: the Veech alternative

Ya. B. Vorobets

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§1. Introduction

In various problems in geometry and dynamics there naturally appears a certain geometric object on a two-dimensional surface which, depending on the context, is called a planar structure, a quadratic differential, or a measured foliation.

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Measured foliations appear in the study of diffeomorphisms and foliations on surfaces, quadratic differentials are one of the objects of Teichmüller theory, and planar structures are closely connected with billiards in rational polygons and interval exchange transformations.

In the present article we will discuss planar structures. A planar structure on a two-dimensional compact orientable surface is a metric of zero curvature having finitely many singular points, at each of which it has a conical singularity with angle that is a multiple of 2π. The progress in the study of such structures during the last 10–15 years is connected mainly with the names of W. Veech and H. Masur and has been achieved by an extensive use of methods from Teichmüller theory. Here, the planar structures themselves occur as a convenient geometric representation for quadratic differentials.

The present paper is devoted to a remarkable property of planar structures first noted by Veech [1]. We consider the simplest example of a planar structure: the torus $\mathbb{R}^2/\mathbb{Z}^2$. In this case the trajectories of the geodesic flow behave particularly simply: each of them is either periodic or uniformly distributed. It turns out that there are many examples of more complicated planar structures having this property (which will be called the Veech alternative in the sequel). More precisely, such are the planar structures having a rich group of affine symmetries. A large number of examples arise in the study of billiard flows in rational polygons, and in this case the Veech alternative provides detailed information on the dynamical properties of the corresponding flows.

The author has made an attempt to circumvent the use of methods from Teichmüller theory, making planar structures the subject of independent investigation. In this way a new proof has been obtained of Veech's theorem [1], establishing the above-mentioned alternative. Furthermore, a new proof of a lemma of Masur [10], Theorem 1.1, has been obtained. This lemma (and the weaker assertion from [4] known before it) plays an important role in practically all investigations in this field. Finally, by developing a geometric approach to the study of planar structures it has become possible to obtain numerous results and examples related to the Veech alternative.

The structure of the paper is as follows. In §2 we give detailed preliminary information related to planar structures, billiards in rational polygons and interval exchange transformations, and also establish a link between these objects. In §3 we give a new proof of Veech's theorem and a new proof of Masur's lemma. We also consider the problem of the distribution of the periodic trajectories of a planar structure. In §4 we give numerous examples of Veech's theorem, both those found by Veech himself and new ones. In §5 we consider covers of planar structures. The results obtained here substantially extend the list of examples of Veech's theorem. Finally, in §6 we derive certain geometric properties of planar structures. A detailed study of these leads to a generalization of Veech's theorem.

The idea of removing from the study of dynamical properties of billiards in rational polygons the non-constructivity connected with the use of Teichmüller theory is due to A. M. Stëpin. The author thanks him for his universal support. The author also thanks D. V. Anosov for stimulating discussions.

The results of this paper have been announced in [18].

§2. Def

2.1. Planar structure

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§2. Definitions and preliminary information

2.1. Planar structures.

Definition 2.1. Let M be a compact, connected, orientable surface. A *planar structure on* M is an atlas ω, consisting of charts of the form (U, f), where U is a domain on M and f is a homeomorphism from U to a domain in $\mathbb{R}^2$, such that the following conditions hold:

(i) the domains U cover the whole surface M except for finitely many points $x_1, \dots, x_k$, called *singular*;
(ii) all coordinate changing functions are shifts in $\mathbb{R}^2$;
(iii) the atlas ω is maximal with respect to (i) and (ii);
(iv) for each singular point x_i there are a punctured neighbourhood $\dot{U}_i$, not containing other singular points, and a map f_i from this neighbourhood to a punctured neighbourhood $\dot{V}$ of a point in $\mathbb{R}^2$ that is a shift in the local coordinates from ω and is such that each point in $\dot{V}$ has exactly m_i preimages. This number m_i is called the *multiplicity* of the singular point x_i.

A singular point of multiplicity 1 is called *removable* (one can then find a planar structure $\widetilde{\omega} \supset \omega$ in which this point is non-singular).

Using the charts of the atlas ω we can lift the Euclidean metric of $\mathbb{R}^2$ to a Riemannian metric of zero curvature on $M \setminus \{x_1, \dots, x_k\}$. At a singular point x_i this metric has a conical singularity with angle $2\pi m_i$, that is, a neighbourhood of this point is isometric to a neighbourhood of the origin in $\mathbb{R}^2$ with a metric that has the form $ds^2 = dr^2 + (m_i\, r\, d\theta)^2$ in polar coordinates (r, θ). By (ii), the holonomy group of this metric is trivial. The area element μ corresponding to this metric is called the *Lebesgue measure* on M.

The metric on M induces the geodesic flow $\{T^t\}$. We restrict our consideration of this flow to a level surface of the energy corresponding to motion with unit velocity. Then the phase space of the flow is a direct product, $\Phi = M \times S^1$. Here, S^1 is the circle of unit directions in $\mathbb{R}^2$ and can be identified with the space of unit tangent vectors at an arbitrary non-singular point. The geodesic flow is well defined on an element of Φ if the trajectory corresponding to the element (a geodesic curve) does not pass through a singular point. Such elements form a set of full measure $\mu \times \lambda$ (with λ the Lebesgue measure on S^1). The geodesic flow preserves the measure $\mu \times \lambda$.

The phase space Φ fibers into invariant surfaces $M \times \{\overline{v}\}$, which are homeomorphic to M. Thus, the restriction of the geodesic flow to an invariant surface can be viewed as a flow on M: the flow with unit velocity in the direction $\overline{v}$. This flow is defined on a subset of M (depending on $\overline{v}$) of full Lebesgue measure and preserves this measure.

Definition 2.2. A flow on M in the direction $\overline{v}$ is called *strongly ergodic* if the Lebesgue measure is the unique (up to scalar multiples) finite Borel measure on M that is invariant under the flow.

In the present case, strong ergodicity is also known as 'unique ergodicity'.

It has been shown by Kerckhoff, Masur and Smillie [5] that for almost all $\overline{v} \in S^1$ the flow in the direction $\overline{v}$ is strongly ergodic. In particular, for the geodesic flow the decomposition into invariant surfaces is also a decomposition into ergodic components (see [11]).

We conclude this subsection by noting that in the study of quadratic differentials there appear planar structures that have a much more complicated form than the ones defined above (see [1], [5]). In fact, for a chart of the atlas ω one allows coordinate transformations of the form $\overline{v} \mapsto \pm\overline{v} + \overline{v}_0$. In this connection, the planar structures in the sense of the above definition are called *orientable*.

The metric on M determined by a non-orientable planar structure has a non-trivial holonomy group. As a result, the geodesic flow on M can, for a certain element $\overline{v}$, only be locally defined; globally it is defined as the (non-orientable) geodesic foliation. The condition of invariance of the Lebesgue measure under the flow is replaced by the condition that it be transversally invariant with respect to the foliation.

A surface with a non-orientable planar structure can be two-sheetedly covered by a surface with an orientable planar structure in a natural manner. The majority of results given below for planar structures can be transferred to the non-orientable case using this cover. Moreover, in the study of billiards in rational polygons only orientable planar structures arise. These are the subject of the next subsection.

2.2. Billiards in rational polygons. Let Q be a polygon in the Euclidean plane $\mathbb{R}^2$, not necessarily convex or simply-connected. A *billiard* in Q is a dynamical system generated by a frictionless motion of a point-ball inside Q with elastic reflections in the boundary ∂Q. The velocity of the ball is taken to be equal to one. The motion is not restricted in time, provided that the ball does not hit a vertex of the polygon. In the opposite case the motion is defined up to the time of hitting the vertex. The phase space $\Phi(Q)$ of this dynamical system can be obtained from the direct product $Q \times S^1$ (where S^1 is the circle of unit directions) by identifying pairs of the form $(q, \overline{v})$ and $(q, \overline{v}')$, where q is a point on a side of Q and $\overline{v}, \overline{v}' \in S^1$ are vectors lying symmetrically with respect to this side. $\Phi(Q)$ inherits from $Q \times S^1$ the measure $\mu \times \lambda$ (with μ the Lebesgue measure on Q and λ the Lebesgue measure on S^1). The billiard flow $\{T_Q^t\}$ is defined for all t in a set of full measure $\mu \times \lambda$; it also preserves the latter measure.

Let a be a side of Q, $\widetilde{r}_a$ the planar symmetry with respect to this side, and r_a the linear part of the operator $\widetilde{r}_a$. Further, let R be the subgroup of $O(2)$ generated by the operators r_a.

Definition 2.3. The polygon Q is called *rational* if R is a finite group. For simply-connected Q this condition is equivalent to all angles being commensurable with π.

A construction of Zemlyakov and Katok [12] reduces a billiard in a rational polygon to the geodesic flow on a certain surface with a planar structure. A version of this construction is given below.

So, let Q be a rational polygon. We set $\widetilde{M} = Q \times R$ and introduce on $\widetilde{M}$ the direct product topology (on R we take the discrete topology). We say that two elements (q_1, r_1) and (q_2, r_2) in $\widetilde{M}$ are *equivalent* if they are equal, or if $q_1 = q_2$ is a point on a side a of Q and $r_1^{-1} r_2 = r_a$, or if $q_1 = q_2$ is a vertex of Q from

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which sides a and b issue and $r_1^{-1}r_2$ belongs to the subgroup of R generated by r_a and r_b. Let M be the quotient space of $\widetilde{M}$ with respect to this equivalence relation, endowed with the quotient topology. It can be readily seen that M is a compact orientable surface, obtained by 'gluing' $|R|$ copies of Q. The polygon Q naturally embeds into M, as $Q \times \{\mathrm{id}\}$. The collection of charts $\{(U_r, f_r)\}_{r\in R}$, with $U_r = (\operatorname{int} Q) \times \{r\}$ and f_r the map from U_r into $\mathbb{R}^2$ defined by $f_r(q,r) = r(q)$, can be extended to a planar structure ω on M in a natural manner. The singular points of this structure correspond to the vertices of Q. In fact, let x be a vertex of Q with interior angle $\pi m/n$, where m/n is a fraction in its lowest terms. For each $r \in R$ the neighbourhood of $(x,r) \in M$ consists of $2n$ copies of Q glued together, and at this point ω has a conical singularity with angle $\pi\dfrac{m}{n}\cdot 2n = 2\pi m$. Thus, x gives rise to $|R|/(2n)$ singular points of multiplicity m.

An arbitrary trajectory of the geodesic flow on M becomes a billiard flow under the natural projection of M onto Q. Conversely, each billiard trajectory in Q 'can be straightened' to a trajectory of the geodesic flow on M.

The group R acts on M ($\widetilde{r}(q,r) := (q,\widetilde{r}r)$) and on the circle S^1, so it acts also on the phase space $\Phi = M \times S^1$ of the geodesic flow on M. This action commutes with the geodesic flow $\{T^t\}$, so there is a well-defined quotient flow $\{T^t\}/R$ on the quotient space Φ/R. We give two representations of this quotient flow.

First, the domain $D_1 = \operatorname{int} Q \times S^1 \subset \Phi$ is a fundamental domain for the action of R on Φ. Also, the points on the boundary ∂D_1 which are identified under the action of R are precisely the points of $Q \times S^1$ that are identified when constructing the phase space of the billiard in Q. Moreover, the billiard flow $\{T_Q^t\}$ and the geodesic flow $\{T^t\}$ act identically on the elements of D_1 up to the time of hitting the boundary ∂D_1. Consequently, the quotient flow $\{T^t\}/R$ is isomorphic to the billiard flow in Q.

Further, let J be an arc of the circle S^1 that is a fundamental domain for the action of R on S^1. The end-points j_1 and j_2 of J are fixed points for certain transformations r_1 and r_2 in R (r_1 and r_2 are reflections; they generate R). Then $D_2 = M \times J$ is a fundamental domain for the action of R on Φ. The boundary ∂D_2 consists of two components, $M \times \{j_1\}$ and $M \times \{j_2\}$. Under the action of R, boundary points of the form (x, j_n) and $(r(x), j_n)$, $x \in M$, $n = 1, 2$, are identified. This implies that the phase space of the quotient flow (or billiard) can be stratified into invariant surfaces all except two of which are isomorphic to invariant surfaces of the geodesic flow, while the two boundary surfaces can be two-sheetedly covered by such surfaces.

2.3. Interval exchange transformations.

Definition 2.4. Suppose we are given an interval $I = [a,b]$ on the real number axis and points $a_1, \dots, a_m$ in it with $a < a_1 < \cdots < a_m < b$. An *interval exchange transformation* is a bijective transformation T of $I \setminus \{a, a_1, \dots, a_m, b\}$ into I that is a shift on each of the intervals (a, a_1), (a_1, a_2), ... , (a_m, b).

Sometimes an interval exchange transformation T is defined on some of the points $a, a_1, \dots, a_m, b$ in such a way that it is left or right continuous at the corresponding points and such that its bijectivity is not violated.

If T is an interval exchange transformation, then all its powers T^n, $n \geqslant 1$, and its inverse T^{-1} are also interval exchange transformations. For all $x \in I$ except for countably many of them, T^n is defined for any $n \in \mathbb{Z}$. For an arbitrary interval exchange transformation T, we let $C(T)$ be the set of points $x \in I$ such that $T^{n_1}x$ and $T^{n_2}x$ are not defined for some $n_1 > 0$, $n_2 < 0$. Clearly, $C(T)$ is a finite set.

Theorem 2.1 [15]. *From the intervals into which I is partitioned by the points of $C(T)$ we can, in a unique manner, form non-intersecting T-invariant sets $K_1, \dots, K_s$, with $I \setminus C(T)$ as union, such that for each i, $1 \leqslant i \leqslant s$, either K_i consists of intervals of the same length that are cyclically permuted by T or the restriction of T to K_i is minimal, that is, the set $\{T^n x\}_{n \geqslant 0}$ is everywhere dense in K_i for any $x \in K_i$ for which $T^n x$ is defined for all $n \geqslant 0$.*

In the first situation we call K_i a *periodic* component of T; in the second situation we call it a *minimal* component. The interval exchange transformation T is said to be *minimal* if it has a single component which is also minimal. Theorem 2.1 implies the following sufficient condition for T to be minimal.

Theorem 2.2 [14]. *If $C(T) = \varnothing$, then T is either the identity or minimal.*

Ergodic properties of interval exchange transformations are described in the following theorem.

Theorem 2.3 [14]. *Any aperiodic (that is, not having periodic components) interval exchange transformation has only finitely many ergodic invariant normalized Borel measures.*

The relation between interval exchange transformations and planar structures is described in the following two constructions.

Let M be a surface with a planar structure ω, and let $\overline{v}$ be some direction. We consider an arbitrary geodesic interval I perpendicular to $\overline{v}$ (I may contain interior singular points and, moreover, its beginning and end may coincide). Let T be the first return map. It induces on I a flow in the direction $\overline{v}$. The map T is defined at $x \in I$ if the trajectory emitted from x in the direction $\overline{v}$ intersects I at a certain non-singular point y; in that case, $Tx = y$.

Proposition 2.4. *The map T is an interval exchange transformation. Moreover, the number of intervals that are exchanged is bounded above by a constant depending on the planar structure only. Any trajectory in the direction $\overline{v}$ and emitted from a point $x \in I$ returns to I or hits a singular point in a time span bounded above by a constant that is independent of x.*

Proof. Let $x_1, \dots, x_m$ be the points in I such that the trajectory emitted from x_i in the direction $\overline{v}$ hits a singular point and does not return to I. Clearly, the number of such points does not increase the sum of the multiplicities of all singular points of the planar structure. We add to them the (at most two) points that are mapped by T to the end-points of I. Let J be one of the subintervals into which the $x_1, \dots, x_m$ partition I. Poincaré's return theorem easily implies that the trajectories emitted from points of J in the direction $\overline{v}$ return to I and, moreover, do not increase S/j, where S is the area of the surface M and j is the length of J.

Moreover, it is obvious that the restriction of T to J is a shift. Thus, the assertion has been proved completely.

Definition 2.5. A *saddle connection* of a planar structure is a geodesic interval joining two singular points and not having singular points in its interior.

The phrase 'saddle connection' is related to the fact that a singular point of a planar structure is a saddle point for the flow in a definite direction on M. Saddle connections parallel to a direction $\overline{v}$ play, for the flow on M in this direction, the same role as the points of $C(T)$ play for the interval exchange transformation T. The similar object for billiards in rational polygons (a billiard trajectory joining two vertices) is called a *generalized diagonal.*

Proposition 2.5. *The flow in a direction $\overline{v}$ on a surface M can be represented as a special flow under an interval exchange transformation with return time that is constant on each of the exchange intervals.*

Proof. Let I be a geodesic interval on M perpendicular to $\overline{v}$. Let $D(I)$ be the union of all trajectories emitted from interior points of I in the direction $\overline{v}$ or $-\overline{v}$. The proof of Proposition 2.4 implies that $D(I)$ is a domain on M. It is bounded by saddle connections and periodic trajectories parallel to $\overline{v}$. The flow in the direction $\overline{v}$ leaves $D(I)$ invariant, and its restriction to $D(I)$ is a special flow over an interval exchange transformation (on I); moreover, the return time is constant on each exchanged interval. In view of the above, to prove the assertion it suffices to find intervals $I_1, \dots, I_m$, perpendicular to $\overline{v}$, such that the corresponding domains $D(I_1), \dots, D(I_m)$ are non-intersecting and have as union all of M (up to singular points and saddle connections parallel to $\overline{v}$).

We first note that an arbitrary periodic trajectory in the direction $\overline{v}$ is contained in a pencil (or band) of periodic trajectories of a single period; this pencil is bounded by saddle connections. Since the number of saddle connections in the direction $\overline{v}$ is finite (it is at most the sum of the multiplicities of the singular points) and each saddle connection bounds at most two pencils, the number of pencils of periodic trajectories in the direction $\overline{v}$ is finite as well. Let $D_1, \dots, D_m$ be these pencils. Clearly, $D_i = D(I_i)$ for certain intervals $I_1, \dots, I_m$ perpendicular to $\overline{v}$. If the union of these pencils (including their boundaries) is not M, then to complement them we choose an interval I_{m+1} perpendicular to $\overline{v}$. The domains $D(I_1), \dots, D(I_{m+1})$ do not contain singular points. If the union of their closures is still not M, to complement them we choose another interval I_{m+2} perpendicular to $\overline{v}$, and so on. The domains thus obtained are bounded by saddle connections parallel to $\overline{v}$ (and each saddle connection bounds at most two domains), therefore there cannot be arbitrarily many of them. Consequently, $M = \bigcup_{1 \leqslant i \leqslant m+n} D(I_i)$ for some $n \geqslant 0$, as required.

Propositions 2.4 and 2.5 (and their proofs) make it possible to obtain analogues of Theorems 2.1, 2.2 and 2.3 for flows.

Theorem 2.6. *Let $D_1, \dots, D_m$ be the domains into which the surface M partitions after deleting singular points and saddle connections parallel to $\overline{v} \in S^1$. Then each D_i is invariant under the flow on M in the direction $\overline{v}$ and either it is a pencil of* *periodic trajectories parallel to $\overline{v}$ or the restriction of the flow to D_i is minimal. The number of domains is bounded by a constant not depending on the direction.*

Theorem 2.7. *If a planar structure ω does not allow saddle connections parallel to a direction $\overline{v}$, then either the flow on M in this direction is minimal or ω does not have singular points and the whole surface M consists of a single pencil of periodic trajectories parallel to $\overline{v}$.*

Theorem 2.8. *If the flow on a surface M in a direction $\overline{v}$ is aperiodic, then there are only finitely many normalized Borel measures on M that are invariant and ergodic with respect to this flow.*

We will now describe a second construction relating interval exchange transformations and planar structures. Let $I = [a, b]$ be an interval on the real number axis, T an interval exchange transformation on I, $a_0 = a, a_1, \dots, a_n = b$ the points of I at which T is not defined, and $b_0 = a, b_1, \dots, b_m = b$ the points at which T^{-1} is not defined. We put $\widetilde{M} = I \times [0, 1]$. By S we denote the subset of $\widetilde{M}$ consisting of the points $(a_i, 1)$, $1 \leqslant i \leqslant n$, and $(b_i, 0)$, $1 \leqslant i \leqslant m$. We identify intervals on the boundary of $\widetilde{M}$: we identify the points (a, t) and (b, t), $t \in (0, 1)$, and we identify $(x, 1)$ and $(Tx, 0)$ for $x \in I$ with Tx defined. Further, we identify the points of S that are identical end-points of identified intervals (that is, simultaneously upper or lower for vertical intervals, left or right for horizontal intervals). We denote by M the quotient space of $\widetilde{M}$ corresponding to the above identification. Then M is a compact orientable surface. The chart (U, id) with $U = (a, b) \times (0, 1)$ can be completed in a natural manner to a planar structure ω on M. The singular points of ω correspond to the points of S. By construction, the flow on M in the vertical direction is a special flow over T with return time equal to 1. In particular, this implies that the above-described construction cannot yield every planar structure.

There are other ways of associating a planar structure with an interval exchange transformation (see, for example, [4], §3).

2.4. Elementary planar structures.

Theorem 2.9. *Let ω be a planar structure on a surface M, let $m_1, \dots, m_k$ be the multiplicities of its singular points, and let χ be the Euler characteristic of M. Then*

$$-\chi = \sum_{i=1}^{k} (m_i - 1).$$

Proof. Both parts of this formula do not change when singular points are added or deleted, so we may assume that ω has singular points.

Lemma 2.10 [5]. *If ω has singular points, then every set of pairwise non-intersecting (that is, without common interior points) saddle connections can be complemented to a triangulation of M whose vertices are singular points, whose edges are saddle connections, and whose faces are triangles not containing singular points in their interior (ω-triangles, see §6).*

Let v, e, f be the numbers of vertices, edges and faces of an arbitrary triangulation of M such as described in Lemma 2.10. Clearly, $v = k$, $3f = 2e$, so Euler's formula $\chi = v - e + f$ implies that $-\chi = f/2 - k$. The sum of all angles of all faces in the triangulation is equal to πf. On the other hand, it is equal to the sum of the full angles at the singular points of ω, that is, $\sum_{i=1}^{k} 2\pi m_i$. Hence: $f/2 = \sum_{i=1}^{k} m_i$ and $-\chi = f/2 - k = \sum_{i=1}^{k}(m_i - 1)$, as required.

In particular, the proof of the theorem implies that there do not exist planar structures on the sphere. Planar structures on a torus can have removable singularities only. On the other hand, a planar structure on a surface of genus $g > 1$ must have at least one non-removable singularity.

Examples of planar structures on a torus are well known. Let $\overline{v}_1$, $\overline{v}_2$ be linearly independent vectors in $\mathbb{R}^2$. By $\mathbb{T}_{\overline{v}_1,\overline{v}_2}$ we denote the quotient space of the group $\mathbb{R}^2$ by the subgroup $\mathbb{Z}\overline{v}_1 \oplus \mathbb{Z}\overline{v}_2$. Then $\mathbb{T}_{\overline{v}_1,\overline{v}_2}$ is a torus, the canonical projection $\pi\colon \mathbb{R}^2 \to \mathbb{T}_{\overline{v}_1,\overline{v}_2}$ is a local homeomorphism, and the continuous maps from domains in $\mathbb{T}_{\overline{v}_1,\overline{v}_2}$ into $\mathbb{R}^2$ that are right inverse to π define on $\mathbb{T}_{\overline{v}_1,\overline{v}_2}$ a planar structure without singular points. This structure is called a *planar torus*. It will turn out that every planar structure on a torus can be obtained in this manner.

Definition 2.6. Let (M_1, ω_1) and (M_2, ω_2) be surfaces with planar structures. A homeomorphism $f\colon M_1 \to M_2$ is called an *isomorphism* of planar structures ω_1 and ω_2 if it maps the singular points of ω_1 to the singular points of ω_2 and is a shift in the local coordinates of ω_1 and ω_2.

Proposition 2.11. *An arbitrary planar structure on a torus is isomorphic to a planar torus to which are added finitely many removable singular points.*

Proof. By Theorem 2.9 it suffices to prove that a planar structure ω without singular points on a torus M is isomorphic to a planar torus. We will first show that ω has a periodic trajectory. Let $A \in M$ and let I be a geodesic interval starting at A in an arbitrary direction $\overline{u}_1$. The trajectory emitted from A in a direction $\overline{u}_2$ perpendicular to $\overline{u}_1$ intersects I at a point A'. We denote by s_1, s_2 the distances to be travelled along I and along the trajectory, respectively, from A to A'. As can be readily seen, the trajectory emitted from A in a direction $\overline{e}_1$ parallel to $-s_1\overline{u}_1 + s_2\overline{u}_2$ is periodic. By Theorem 2.7, the whole surface M is a single pencil of periodic trajectories in the direction $\overline{e}_1$, having the same length l_1. We draw the geodesic interval J in a direction $\overline{e}_2$ perpendicular to $\overline{e}_1$ whose length is the width w_1 of the pencil of periodic trajectories in the direction $\overline{e}_1$. The end-points A and A'' of this interval belong to the same trajectory of the pencil; all other trajectories intersect J just once. Let l_2 be the distance from A'' to A when moving along the direction $\overline{e}_1$. Then the trajectories parallel to $\overline{v}_2 = w_1\overline{e}_2 + l_2\overline{e}_1$ form a pencil of periodic trajectories of length $|\overline{v}_2|$. The trajectories from the pencils parallel to $\overline{e}_1$ and $\overline{v}_2$ intersect one another just once. This implies that the planar structure ω is isomorphic to the planar torus $\mathbb{T}_{l_1\overline{e}_1,\overline{v}_2}$.

It is well known that the flows on $\mathbb{T}_{\overline{v}_1,\overline{v}_2}$ in directions parallel to vectors in $\mathbb{Z}\overline{v}_1 \oplus \mathbb{Z}\overline{v}_2$ are periodic, while the flows in all other directions are strongly ergodic (see, for example, [11]). Comparing this with Theorems 2.6 and 2.8, we conclude that this behaviour is the simplest kind for the geodesic flow on a surface with a planar structure.

Definition 2.7. A planar structure ω on a surface M is said to be *elementary* if the flow on M in an arbitrary direction is either strongly ergodic or has only periodic components. (Here, parallel periodic trajectories from distinct pencils can have incommensurate lengths, that is, the flow in such a direction cannot, in general, be periodic.)

Each trajectory of the geodesic flow on M corresponding to an elementary planar structure either hits a singular point, or is periodic, or is uniformly distributed with respect to the Lebesgue measure on M. Similarly, if the planar structure corresponding to a rational polygon Q is elementary, then every billiard trajectory in Q not hitting a vertex is either periodic or uniformly distributed inside Q.

Results from [9] imply that, in contrast to planar structures on a torus (which are all elementary), elementary planar structures on surfaces of genus $g > 1$ are rare: for a *typical* (see [9]) planar structure the set of directions whose flows are minimal but not strongly ergodic (or even non-ergodic with respect to the Lebesgue measure) has positive Hausdorff dimension. On the other hand, for an arbitrary planar structure, the flows in almost all directions are strongly ergodic (see §2.1). Moreover, there are directions whose flows have a periodic component (according to Masur [6], such directions densely fill S^1). An elementary proof of the existence of periodic trajectories for planar structures corresponding to rational polygons is due to Stëpin (see [13]).

§3. The Veech alternative

In this section we will prove Veech's theorem [1], which gives a sufficient condition for a planar structure to be elementary. Basically, the proof follows the original lines proposed by Veech, but some individual steps are made simpler. Moreover, we will show that Masur's lemma (Lemma 3.5), the most important ingredient of the proof, follows from a theorem of Veech on interval exchange transformations.

3.1. The stabilizer of a planar structure. Let ω be a planar structure on a surface M, and L a saddle connection of it. We can find a chart (U, f) in the atlas of ω such that U contains L (without end-points). Then f maps L to an interval in $\mathbb{R}^2$. We orient this interval in the two possible ways, thus obtaining two oppositely pointing vectors, each of which we call a *development* of L. Clearly, a development of L is well defined, that is, does not depend on the choice of the chart (U, f).

We denote by $\mathrm{SC}(\omega)$ the sequence whose terms are the developments of all saddle connections of ω. If a vector $\overline{v} \in \mathbb{R}^2$ is a development of several saddle connections, it occurs in $\mathrm{SC}(\omega)$ the corresponding number of times. The sequence $\mathrm{SC}(\omega)$ is well defined up to the order of its terms.

Proposition 3.1. *Suppose that ω has at least one singular point. Then the directions of the vectors in $\mathrm{SC}(\omega)$ are everywhere dense on the unit circle. Simultaneously, the sequence $\mathrm{SC}(\omega)$ does not have limit points.*

Proof. For an arbitrary $\overline{v} \in S^1$ and $\varepsilon > 0$ we have to prove that $\mathrm{SC}(\omega)$ contains a vector making with $\overline{v}$ an angle less than ε. Let A be a singular point of the planar structure, and let AA' be a geodesic interval of length $s > 0$ perpendicular to $\overline{v}$. We draw the trajectory L from A in the direction $\overline{v}$. If it is a saddle connection there is nothing to prove. Otherwise L intersects AA' infinitely many times. Let A_n be the point of nth intersection, and let l_n, s_n be the distances from A to A' measured along L and along the interval, respectively. We denote by $\overline{v}_\delta$ the direction forming an angle δ $(0 < \delta < \pi/2)$ with $\overline{v}$ and an obtuse angle with the direction of AA'. For small δ, the trajectory L_δ emitted from A in the direction $\overline{v}_\delta$ intersects AA' after a time $l_n/\cos\delta$ at the point B_δ lying at a distance $s_n - l_n \tan\delta$ from A. This situation persists for increasing δ until either L_δ suddenly becomes larger than $l_n/\cos\delta$ or B_δ merges with A (for $\delta = \arctan(s_n/l_n)$. In any case we obtain a saddle connection forming with $\overline{v}$ an angle not exceeding $\arctan(s_n/l_n)$. Since $s_n \leqslant s$ and $\lim_{n\to\infty} l_n = +\infty$, we find that $\arctan(s_n/l_n) < \varepsilon$ for n large, as required.

We will now prove that $\mathrm{SC}(\omega)$ does not have limit points. Since the number of saddle connections with identical developments does not exceed the sum of the multiplicities of the singular points of ω, it suffices to prove that $\mathrm{SC}(\omega)$ does not have accumulation points. Each singular point has a neighbourhood not containing other singular points, so the length of an arbitrary element in $\mathrm{SC}(\omega)$ is at least some $\varepsilon > 0$. Let $\overline{v} \in \mathbb{R}^2$, $\overline{v} \neq 0$. From all singular points we draw all possible geodesic intervals in the direction $\overline{v}$ of length $|\overline{v}|$ (the intervals may have interior singular points). The number of such intervals is finite (and depends on $\overline{v}$). Compactness considerations imply the existence of a $\delta > 0$ such that the trajectory emitted from an arbitrary point of any such interval in a direction perpendicular to $\overline{v}$ hits a singular point not before time δ. Diminishing δ, if necessary, we may also assert that the punctured δ-neighbourhoods of the end-points of the drawn intervals do not contain singular points. As can be readily seen, the punctured neighbourhood of the vector $\overline{v}$ does not contain members of $\mathrm{SC}(\omega)$. Since $\overline{v}$ is arbitrary, the assertion has been proved.

The above assertion implies, in particular, the existence of a shortest saddle connection. We denote its length by $m(\omega)$.

If $\omega = \{(U_\alpha, f_\alpha)\}_{\alpha\in\mathcal{A}}$ is a planar structure on a surface M and a is a linear invertible operator in $\mathbb{R}^2$, then the atlas $\{(U_\alpha, a \circ f_\alpha)\}_{\alpha\in\mathcal{A}}$ is also a planar structure on M; we denote it by $a\omega$. The planar structures ω and $a\omega$ have the same singular points with the same multiplicities, and also the same saddle connections. If $\overline{v}$ is a development of a saddle connection of ω, then $a\overline{v}$ is a development of the same saddle connection with respect to $a\omega$. In particular, $\mathrm{SC}(a\omega) = a(\mathrm{SC}(\omega))$.

Proposition 3.2. *The function* $d\colon \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}$, $d(a) = m(a\omega)$, *is continuous. Its restriction to the subgroup* $\mathrm{SL}(2,\mathbb{R})$ *is bounded.*

Proof. Let $a, b \in \mathrm{GL}(2,\mathbb{R})$ and let $\overline{v}$ be a development of the shortest saddle connection of $a\omega$. Then $(ba^{-1})\overline{v} \in \mathrm{SC}(b\omega)$, whence

$$m(b\omega) \leqslant |(ba^{-1})\overline{v}| \leqslant \|ba^{-1}\| \cdot |\overline{v}| = \|ba^{-1}\| \cdot m(a\omega),$$

where $\|\cdot\|$ is the Euclidean operator norm. Interchanging the roles of a and b in this inequality, we obtain a second inequality:

$$\|ab^{-1}\|^{-1} \cdot m(a\omega) \leqslant m(b\omega) \leqslant \|ba^{-1}\| \cdot m(a\omega).$$

As $b \to a$, the quantities $\|ab^{-1}\|$ and $\|ba^{-1}\|$ tend to 1, therefore $m(b\omega)$ tends to $m(a\omega)$, that is, d is a continuous function.

For any $a \in \mathrm{SL}(2,\mathbb{R})$ the planar structures $a\omega$ and ω give the same Lebesgue measure on the surface. Consequently, we will have proved that d is bounded on $\mathrm{SL}(2,\mathbb{R})$ if we can prove that $m(\omega) \leqslant \sqrt{2S}$, where S is the area of the surface with planar structure ω. Let A be a singular point of ω, and $\overline{v}$ an arbitrary direction. Starting at A we draw the geodesic interval I of length $l = \sqrt{S}$ in the direction $\overline{v}$. If I hits a singular point, there is nothing to prove. If not, we draw from each point of the interval the trajectory in a direction $\overline{u}$ perpendicular to $\overline{v}$. After a time t not exceeding $S/l = \sqrt{S}$, some such trajectory intersects I again. If there is still no trajectory that has hit a singular point, then at time t either some trajectory hits A or I intersects the trajectory emitted from A. Thus, if we draw from the points of I the trajectories in the directions $\overline{u}$ and $-\overline{u}$, then at some time not exceeding t some such trajectory hits a singular point B. Joining A and B we obtain a saddle connection whose projections on $\overline{v}$ and $\overline{u}$ do not exceed $\sqrt{S}$ and whose length is thus not greater than $\sqrt{2S}$. So $m(\omega) \leqslant \sqrt{2S}$, as required.

Definition 3.1. An *affine automorphism* of a planar structure ω is a homeomorphism $f\colon M \to M$ that maps singular points to singular points and is an affine map in the local coordinates of the atlas of ω. This is equivalent to the fact that f is an isomorphism of the planar structures $a\omega$ and ω, where $a \in \mathrm{GL}(2,\mathbb{R})$. The operator a is said to be the *linear part* of the automorphism f.

Since the areas of surfaces with isomorphic planar structures are equal, and since under transition from ω to $a\omega$ surface area is multiplied by $|\det a|$, the determinant of the linear part of an affine automorphism must be equal to 1 or -1.

Definition 3.2. The *stabilizer* $\Gamma(\omega)$ of a planar structure ω is the set of operators $a \in \mathrm{SL}(2,\mathbb{R})$ for which the planar structure $a\omega$ is isomorphic to ω.

In view of the above, the stabilizer consists of the linear parts of the affine automorphisms preserving the orientation of the surface.

Proposition 3.3. *$\Gamma(\omega)$ is a discrete non-uniform subgroup of the group* $\mathrm{SL}(2,\mathbb{R})$.

Proof. If the planar structures ω_1 and ω_2 are isomorphic, then for any $a \in \mathrm{GL}(2,\mathbb{R})$ the planar structures $a\omega_1$ and $a\omega_2$ are also isomorphic (the isomorphism is given by the same map). This implies that $\Gamma(\omega)$ is a group. Further, since $\mathrm{SC}(a\omega) = \mathrm{SC}(\omega)$ for all $a \in \Gamma(\omega)$, Proposition 3.1 implies that $\Gamma(\omega)$ is a discrete group. Finally, let $\overline{v}$ be a development of an arbitrary saddle connection of the planar structure, and let a be an operator in $\mathrm{SL}(2,\mathbb{R})$ such that $a\overline{v} = 1/2\overline{v}$. Clearly, $m(a^n\omega) \to 0$ as $n \to \infty$. If $\Gamma(\omega)$ were a uniform subgroup, that is, the quotient group $\mathrm{SL}(2,\mathbb{R})/\Gamma(\omega)$ were compact, then some subsequence $\{a^{n_i}\Gamma(\omega)\} \subset \mathrm{SL}(2,\mathbb{R})/\Gamma(\omega)$ would converge to a certain $b\Gamma(\omega)$, with $b \in \mathrm{SL}(2,\mathbb{R})$, as $i \to \infty$. Then there would be a sequence $\{\gamma_i\} \subset \Gamma(\omega)$ such that $a^{n_i}\gamma_i \to b$ in $\mathrm{SL}(2,\mathbb{R})$ as $i \to \infty$. By Proposition 3.2, $m(a^{n_i}\gamma_i\omega) \to m(b\omega)$ as $i \to \infty$. This however, is impossible, since $m(a^{n_i}(\gamma_i\omega)) = m(a^{n_i}\omega) \to 0$ as $i \to \infty$ and $m(b\omega) \neq 0$. This contradiction shows that $\Gamma(\omega)$ is a non-uniform subgroup of $\mathrm{SL}(2,\mathbb{R})$.

3.2. Veech's theorem

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3.2. Veech's theorem.

Theorem 3.4 (the Veech alternative). *Suppose that the planar structure ω is such that its stabilizer $\Gamma(\omega)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Then ω is an elementary planar structure. Moreover, the pencil in the direction $\overline{v}$ has a single periodic component if and only if $a\overline{v} = \overline{v}$ for some $a \in \Gamma(\omega)$, $a \neq 1$; if ω has singular points, the latter is true if and only if ω has a saddle connection parallel to $\overline{v}$.*

The proof of this theorem consists in the successive application of the following four lemmas.

Lemma 3.5 (Masur [10], Theorem 1.1). *Suppose that the planar structure ω has singular points and is such that $m(g^t\omega) \not\to 0$ as $t \to +\infty$, where*

$$g^t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

Then the flow in the vertical direction is strongly ergodic.

(This formulation of the lemma is different from the original formulation; however, it is completely equivalent to it.)

Lemma 3.6. *If the planar structure ω is such that $m(g^t\omega) \to 0$ as $t \to +\infty$, then $g^t(1 \cdot \Gamma(\omega)) \to \infty$ in $\mathrm{SL}(2,\mathbb{R})/\Gamma(\omega)$ as $t \to +\infty$.*

Proof. We proceed as in the proof of Proposition 3.3. Suppose that $g^t(1\cdot\Gamma(\omega)) \not\to \infty$ as $t \to +\infty$. Then for some sequence $\{t_i\} \subset \mathbb{R}$ tending to $+\infty$ we have $g^{t_i}(1 \cdot \Gamma(\omega)) \to h \cdot \Gamma(\omega)$, where $h \in \mathrm{SL}(2,\mathbb{R})$. Furthermore, we can find a sequence $\{\gamma_i\} \subset \Gamma(\omega)$ such that $g^{t_i}\gamma_i \to h$ in $\mathrm{SL}(2,\mathbb{R})$ as $i \to \infty$. By Proposition 3.2, this implies that $m(g^{t_i}\gamma_i\omega) \to m(h\omega)$ as $i \to \infty$. Since $m(g^{t_i}\gamma_i\omega) = m(g^{t_i}\omega)$ and $m(h\omega) \neq 0$, we find that $m(g^t\omega) \not\to 0$ as $t \to +\infty$. This proves the lemma.

Lemma 3.7. *Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$ such that $g^t(1 \cdot \Gamma) \to \infty$ in $\mathrm{SL}(2,\mathbb{R})/\Gamma$ as $t \to +\infty$. Then $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma$ for some $\alpha \neq 0$.*

Proof. The group $\mathrm{SL}(2,\mathbb{R})$ acts on the Lobachevskii plane $\mathbb{H}^2$, realized as the half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, by linear-fractional transformations:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Moreover (see [16]), for the action of the lattice Γ there is a fundamental polygon $D \subset \mathbb{H}^2$ with finitely many sides. D has finitely many vertices $A_1, \dots, A_k$ on the absolute, and the sides of D through A_i are mapped to each other by an operator $a_i \in \Gamma$ of the form $\pm r_i \sigma r_i^{-1}$, where σ is the operator formed by a fixed horizontal vector (it acts on $\mathbb{H}^2$ by the shift $z \mapsto z+\alpha$) and r_i is the rotation operator mapping ∞ to A_i.

For each $z \in \mathbb{H}^2$ we let $\gamma(z)$ be the element of Γ for which $\gamma(z)z \in D$. The curve $\gamma(g^{-t}z)g^{-t}z$ (which is, in general, discontinuous) does not have limit points as $t \to +\infty$. In fact, otherwise $\gamma(g^{-t_i}z)g^{-t_i}z \to \widetilde{z} \in \mathbb{H}^2$ as $i \to \infty$, where $\{t_i\}$ is a sequence tending to $+\infty$. Since the operators mapping z into a closed neighbourhood of $\widetilde{z}$ form a compact set in $\mathrm{SL}(2,\mathbb{R})$, the sequence $\{\gamma(g^{-t_i}z)g^{-t_i}\}$ would have a limit point in $\mathrm{SL}(2,\mathbb{R})$. Hence, the sequence of inverse elements $\{g^{t_i}(\gamma(g^{-t_i}z))^{-1}\}$ would also have a limit point, contradicting the fact that $g^t(1\cdot\Gamma) \to \infty$ in $\mathrm{SL}(2,\mathbb{R})/\Gamma$ as $t \to +\infty$.

We put $\Pi(c) = \{z \in \mathbb{C} \mid \operatorname{Im} z \geqslant c > 0\}$, $\Pi(c;\alpha,\beta) = \{z \in \Pi(c) \mid \alpha \leqslant \operatorname{Re} z \leqslant \beta\}$, $\widetilde{\Pi}(c;\alpha,\beta) = \{z \in \Pi(c;\alpha,\beta) \mid \operatorname{Im} z = c\}$. The polygon D consists of a compact part K and k *beaks* (or *wedges*) $r_i(\Pi(c_i;\alpha_i,\beta_i))$, $1 \leqslant i \leqslant k$, separated from K by arcs $r_i(\widetilde{\Pi}(c_i;\alpha_i,\beta_i))$ (here, α_i and β_i are uniquely determined by D). For every $z \in r_i(\Pi(c_i))$, $\gamma(z)$ is easily seen to be a power of a_i, and $\gamma(z)z \in r_i(\Pi(c_i;\alpha_i,\beta_i))$. If $z \in r_i(\partial\Pi(c_i))$, then $\gamma(z)z \in r_i(\widetilde{\Pi}(c_i;\alpha_i,\beta_i))$.

We fix a point $z \in \mathbb{H}^2$. Since the curve $\gamma(g^{-t}z)g^{-t}z$ does not have limit points as $t \to +\infty$, it does not intersect K for t larger than some t_0. We put $\widetilde{\gamma} = \gamma(g^{-t_0}z)$; then $\widetilde{\gamma}(g^{-t_0}z) \in r_i(\Pi(c_i;\alpha_i,\beta_i))$, where $1 \leqslant i \leqslant k$. The curve $g^{-t}z$ is a Euclidean straight line, tending, as $t \to +\infty$, to the point 0 on the absolute. Therefore the curve $\widetilde{\gamma}(g^{-t}z)$, $t \geqslant t_0$, is a Euclidean straight line or circle tending to the point $\widetilde{\gamma}(0)$. If $\widetilde{\gamma}(0) \neq A_i$, then at a certain moment of time $t_1 \geqslant t_0$ the curve $\widetilde{\gamma}(g^{-t}z)$ would intersect the boundary of the domain $r_i(\Pi(c_i))$; at the same time the curve $\gamma(g^{-t}z)g^{-t}z$ would intersect the arc $r_i(\widetilde{\Pi}(c_i;\alpha_i,\beta_i))$ and would enter K. This contradicts the choice of t_0. So $\widetilde{\gamma}(0) = A_i$, and then $\widetilde{a} = \widetilde{\gamma}^{-1}a_i\widetilde{\gamma} \in \Gamma$ maps the point 0 to itself. Since $\widetilde{a}$ is conjugate to a_i (and, consequently, also to some $\pm\sigma$), it must have the form $\pm\begin{pmatrix}1 & 0\\ \alpha & 1\end{pmatrix}$, $\alpha \neq 0$. Moreover, $\widetilde{a}^2 = \begin{pmatrix}1 & 0\\ 2\alpha & 1\end{pmatrix} \in \Gamma$. This proves the lemma.

Lemma 3.8. *If* $a = \begin{pmatrix}1 & 0\\ \alpha & 1\end{pmatrix} \in \Gamma(\omega)$, $\alpha \neq 0$, *and the planar structure* ω *has singular points, then the flow in the vertical direction corresponding to* ω *has only periodic components. Moreover, for each pencil of periodic trajectories, the ratio of the length to the width is commensurate with* α.

Proof. An affine automorphism with linear part a acts as a permutation on the finite set $\{L_1,\dots,L_k\}$ consisting of the trajectories emitted from singular points of ω in the vertical direction (upwards and downwards). Hence, for some $n \in \mathbb{N}$ the affine automorphism φ^n maps each trajectory L_i to itself. Moreover, clearly, all points on the trajectory remain fixed. Consequently, the closure of the trajectory does not have interior points (otherwise the linear part of φ^n would be the identity). By Theorem 2.6, L_i is a saddle connection. Now let p be a point on the surface not lying on a vertical saddle connection and L the vertical trajectory passing through p. For some $\varepsilon > 0$ the point p does not belong to the ε-neighbourhood U_ε of the union of the saddle connections L_1, ..., L_k. However, it follows from the above that U_ε is invariant under the flow in the vertical direction, so no point of L can belong to U_ε. Applying Theorem 2.6 we find that L is a periodic trajectory.

So, the flow in the vertical direction splits into periodic pencils. Moreover, under the action of φ^n a point in the pencil at a distance x from its left boundary moves vertically upwards through a distance $x \cdot n\alpha$. Since φ^n is continuous and leaves

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invariant the points on the boundary of the pencil, for a value of x equal to the width w of the pencil the magnitude of the motion must be a multiple of the length l of the pencil. Thus, $w \cdot n\alpha = lm$, $m \in \mathbb{N}$; in particular, α and l/w are commensurate.

Proof of Theorem 3.4. First, since the group of automorphisms (shifts) of the planar torus acts transitively on it, by adding a removable singularity to the planar torus its stabilizer does not change. By Proposition 2.11 we may assume that, without loss of generality, the planar structure ω does have singular points.

Let $\overline{v}$ be an arbitrary direction and b the rotation mapping $\overline{v}$ into the vertical direction. Clearly, $\Gamma(b\omega) = b\Gamma(\omega)b^{-1}$, hence $\Gamma(b\omega)$ is also a lattice. Furthermore, the planar structures ω and $b\omega$ induce the same Lebesgue measure on the surface, and the flow in the vertical direction for $b\omega$ coincides with the flow in the direction $\overline{v}$ for ω. Finally, $\mathrm{SC}(b\omega)$ contains a vertical vector if and only if $\mathrm{SC}(\omega)$ contains a vector parallel to $\overline{v}$, and if $a\overline{v} = \overline{v}$, with $a \in \Gamma(\omega)$, $a \neq 1$, then $a_1(b\overline{v}) = b\overline{v}$, with $a_1 = bab^{-1} \in \Gamma(b\omega)$, $a_1 \neq 1$. Thus, without loss of generality we may assume that the direction $\overline{v}$ mentioned in the conditions of the theorem is vertical. If ω has a vertical saddle connection, then clearly $m(g^t\omega) \to 0$ as $t \to +\infty$ and, by Lemmas 3.6 and 3.7, $\Gamma(\omega)$ must contain a non-identity element mapping the vertical vector to itself. In the presence of such an element, the flow in the vertical direction can have periodic components only, by Lemma 3.8. Finally, if the flow in the vertical direction has a periodic component, then a vertical saddle connection can be found on the boundary of the component. So, to finish the proof it remains to show that the flow in the vertical direction is strongly ergodic if it does not split into periodic components. This follows from Lemmas 3.5–3.8.

We finish this subsection with another lemma, which is in a certain sense the converse of Lemma 3.8 (we will need it in §4).

Definition 3.3. The least common multiple $\mathrm{LCM}(r_1,\dots,r_k)$ of positive rationally commensurate numbers $r_1,\dots,r_k$ is the least positive integer that is an integral multiple of each of these numbers.

Lemma 3.9. *Suppose that the flow in the vertical direction corresponding to the planar structure ω has only periodic components. If $r_1,\dots,r_k$, the ratios of the length to the width of all pencils in the flow, are commensurate, then* $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma(\omega)$, *where* $\alpha = \mathrm{LCM}(r_1,\dots,r_k)$.

Proof. We define a bijective map φ from the surface onto itself as follows: φ leaves invariant each point p not belonging to the vertical saddle connection; otherwise p is moved vertically upwards through a distance $x \cdot \alpha$, where x is the distance from p to the left boundary of the pencil of vertical periodic trajectories containing p. Since α is an integral multiple of each of the numbers $r_1,\dots,r_k$, it follows that φ is a homeomorphism. By construction, φ is then an affine automorphism of ω with linear part $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$.

3.3. Masur's lemma. Let T be an interval exchange transformation on the interval $I = [a, b]$ (see §2.3) and $a_0 = a, a_1, \dots, a_k = b$ the points in I at which T is not defined. We denote by $\varepsilon(T)$ the length of the shortest intervals into which I is partitioned by these points $a_0, a_1, \dots, a_k$. Furthermore, for each $n \in \mathbb{N}$ we put $\varepsilon_n(T) = \varepsilon(T^n)$.

Theorem 3.10 [3], [15]. *If $n\varepsilon_n(T) \nrightarrow 0$ as $n \to \infty$ and T is minimal, then T is strongly ergodic.*

Now let ω be the planar structure on the surface M, and let $\overline{v}$ be an arbitrary direction. For each $l > 0$ we consider its saddle connections of length at most l and not parallel to $\overline{v}$, and project these onto the direction orthogonal to $\overline{v}$. We denote by $E_l(\overline{v})$ the shortest of these projections.

Theorem 3.11. *If $lE_l(\overline{v}) \nrightarrow 0$ as $l \to +\infty$ and the flow on M in the direction $\overline{v}$ is minimal, then this flow is strongly ergodic.*

This theorem follows from the following proposition.

Proposition 3.12. *Theorems 3.10 and 3.11 are equivalent. Lemma 3.5 follows from Theorem 3.11.*

Proof. We will first prove that Theorem 3.11 follows from Theorem 3.10. We suppose that the flow on M in the direction $\overline{v}$ is minimal and that $lE_l(\overline{v}) \nrightarrow 0$ as $l \to +\infty$. We consider an arbitrary geodesic interval I perpendicular to $\overline{v}$. By Proposition 2.4 and Theorem 2.6, the flow in the direction $\overline{v}$ induces on I a minimal interval exchange transformation T and is a special flow over T. The set S of points of I from which a trajectory issues in the direction $\overline{v}$ and hits a singular point, is everywhere dense in I (this follows from Theorem 2.6). By making I smaller we may assume that its end-points are contained in S. Let S_1 be the set of points of S for which the trajectory in the direction $\overline{v}$ hits a singular point before the first return to I, returns to an end-point of I, or is an end-point of I itself. The set S_1 is finite, and $I \setminus S_1$ consists of finitely many intervals on each of which T acts as a shift. Consequently, we may assume that T is not defined at the points of S_1. Moreover, the trajectory in the direction $\overline{v}$ emitted from a point $x \in I$ passes through a singular point before a time t_1 if $x \in S_1$, and returns to I before a time t_2 if $x \notin S_1$ (with t_1, t_2 certain constants).

Let $n \in \mathbb{N}$, and let x_1, x_2 be points of I at which T^n is not defined and with mutual distance $\varepsilon_n(T)$. The trajectories in the direction $\overline{v}$ emitted from x_1 and x_2 hit singular points lying at distance not exceeding $(n-1)t_2 + t_1$. This implies the existence of a saddle connection whose projection onto the direction $\overline{v}$ has length at most $(n-1)t_2 + t_1$, while its projection onto the orthogonal direction is positive and has length at most $\varepsilon_n(T)$. The length of this saddle connection does not exceed $2nt_2$ for n large. Consequently, $E_{2nt_2}(\overline{v}) \leqslant \varepsilon_n(T)$ for n large. So, $n\varepsilon_n(T) \nrightarrow 0$ as $n \to \infty$, and T is strongly ergodic by Theorem 3.10. In this case the flow on M in the direction $\overline{v}$ is also strongly ergodic, being a special flow over T.

Now we will, conversely, derive Theorem 3.10 from Theorem 3.11. Let T be a minimal interval exchange transformation on I and assume that $n\varepsilon_n(T) \nrightarrow 0$ as $n \to \infty$. We associate with T a planar structure on a certain surface M, as described in §2.3. The interval I embeds into M as a horizontal interval, and the

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flow on M in the vertical direction $\overline{v}$ is a special flow over T, hence minimal. We fix an $n \in \mathbb{N}$ and consider a saddle connection L of length not exceeding n and with horizontal projection $E_n(\overline{v})$. We denote the end-points of L by A_1 and A_2, in such a way that the direction from A_1 to A_2 makes an acute angle with $\overline{v}$. The trajectory emitted from A_1 in the direction $-\overline{v}$ hits after time 1 a point $x_1 \in I$ at which T is not defined. Let $x_2 \in I$ be the point of I nearest to x_1 with the property that the trajectory emitted from it in the direction $\overline{v}$ passes through a singular point lying at a distance not exceeding $n+1$. Clearly, T^n is not defined at x_2. Hence $\varepsilon_n(T)$ is at most equal to the distance between x_1 and x_2, which, by the choice of L, does not exceed $E_n(\overline{v})$. So, $\varepsilon_n(T) \leqslant E_n(\overline{v})$, that is, $lE_l(\overline{v}) \nrightarrow 0$ as $l \to +\infty$. By Theorem 3.11, the flow in the direction $\overline{v}$ is strongly ergodic, and hence its generating map T is strongly ergodic.

Finally, we will show that Lemma 3.5 is a consequence of Theorem 3.11. Clearly, the condition $m(g^t\omega) \to 0$ as $t \to +\infty$ is equivalent to the fact that for an arbitrary $\delta > 0$ and sufficiently large l there is a saddle connection whose vertical projection has length at most δl, while the length of its horizontal projection is at most δ / l. It is also equivalent to the fact that for sufficiently large l there is a saddle connection whose vertical projection has length at most l, while the length of its horizontal projection is at most δ^2/l. The condition $\lim_{l\to+\infty} lE_l(\overline{v}) = 0$, where $\overline{v}$ is the vertical direction, differs from the previous condition only by an additional restriction in the choice of the saddle connections: they cannot be vertical. Thus, the condition $m(g^t)\omega \nrightarrow 0$ as $t \to +\infty$ implies that $lE_l(\overline{v}) \nrightarrow 0$ as $l \to +\infty$. Moreover, in this case the planar structure ω does not have vertical saddle connections and, by Theorem 2.7, the flow in the vertical direction is minimal. Thus, all conditions in Theorem 3.11 hold.

It follows from the above proof that, for a flow on M in the direction $\overline{v}$, Theorem 3.11 asserts more than Lemma 3.5 if and only if ω allows saddle connections parallel to $\overline{v}$.

3.4. Pencils of periodic trajectories. Let ω be a planar structure on a surface M, having singular points. We assume that the geodesic trajectory emitted from a singular point $x \in M$ in a direction $\overline{v} \in S^1$ returns to x after a time t. Then each of the vectors $t\overline{v}$ and $-t\overline{v}$ is called a *development* of this periodic trajectory. The points on the surface from which periodic trajectories with such developments leave (or, which is the same, the trajectories themselves) form a *pencil*: a domain bounded by saddle connections in the direction $\overline{v}$ (see §2.3). Moreover, there can be several pencils corresponding to the same development. A pencil is called *multiple* if its trajectories are periodic trajectories of minimal length, traversed several times.

For each $R > 0$ we denote by $N(R)$ and $S(R)$ the number and the sum of the areas of the pencils of periodic trajectories of length at most R, without counting multiple pencils. Let $N^*(R)$ and $S^*(R)$ be the same quantities, but including multiple pencils in the count. Moreover, we denote by $N_0(R)$ the number of saddle connections of ω whose lengths are at most R. Masur [7], [8] has shown that for large R:

$$c_- \cdot R^2 \leqslant N_0(R) \leqslant c_+ \cdot R^2, \qquad c_0 \cdot N_0(R) \leqslant N(R) \leqslant N_0(R),$$

where c_+, c_-, c_0 are positive constants. Since $N^*(R) = N(R) + N(R/2) + N(R/3) + \cdots$, for large R we also have

$$c_-^* \cdot R^2 \leqslant N^*(R) \leqslant c_+^* \cdot R^2,$$

where c_+^*, c_-^* are positive constants. Moreover, we clearly have $S(R) \leqslant S \cdot N(R)$, $S^*(R) \leqslant S \cdot N^*(R)$, where S is the surface area of M.

The following theorem supplements Veech's result on the distribution of periodic trajectories of planar structures, obtained in [1].

Theorem 3.13. *If the stabilizer $\Gamma(\omega)$ is a lattice, then each of the quantities $N_0(R)$, $N(R)$, $S(R)$, $N^*(R)$, $S^*(R)$ has, as $R \to \infty$, asymptotic behaviour of the form $cR^2 + o(R^2)$, where c is a positive constant.*

Proof. Since $\Gamma(\omega)$ is a lattice, we can find finitely many operators $a_1, \dots, a_k \in \Gamma(\omega)$ and non-zero vectors $\overline{v}_1, \dots, \overline{v}_k$, where $a_i\overline{v}_i = \pm\overline{v}_i$ but $a_i \neq \pm 1$, such that every operator $a \in \Gamma(\omega)$, $a \neq \pm 1$, having eigenvector $\overline{v}$ with eigenvalues ± 1 is conjugate in $\Gamma(\omega)$ to an operator of the form $\pm a_i^n$, $1 \leqslant i \leqslant k$, $n \neq 0$ (see [16]). Moreover, $\overline{v}$ is collinear with a vector of the form $\gamma\overline{v}_i$, $\gamma \in \Gamma(\omega)$. Without loss of generality we may assume that the vector $\overline{v}_i$ in this representation is uniquely determined. By Theorem 3.4, the vectors parallel to saddle connections (or periodic trajectories) of ω are precisely the fixed vectors of the non-identity operators in $\Gamma(\omega)$, that is, vectors of the form $\gamma\overline{v}_i$, $1 \leqslant i \leqslant k$, $\gamma \in \Gamma(\omega)$, and vectors collinear with these. Let $l_1^i, \dots, l_{m_i}^i$ be the lengths of the saddle connections parallel to $\overline{v}_i$, let $L_1^i, \dots, L_{n_i}^i$ be the lengths of the pencils of periodic trajectories parallel to $\overline{v}_i$, and let $S_1^i, \dots, S_{n_i}^i$ be the areas of these pencils. If $\gamma \in \Gamma(\omega)$ and φ is an affine automorphism of ω with linear part γ, then φ is a bijective correspondence between the saddle connections and the pencils of periodic trajectories parallel to $\overline{v}_i$ and $\gamma\overline{v}_i$, respectively. Under this correspondence, the lengths of the saddle connections and pencils are multiplied by $|\gamma\overline{v}_i|/|\overline{v}_i|$, while the areas of the pencils are not changed. Thus, from what was said above we obtain

$$N_0(R) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} N_{\overline{v}_i}(R \cdot |\overline{v}_i|/l_i), \qquad N(R) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} N_{\overline{v}_i}(R \cdot |\overline{v}_i|/L_i),$$

$$S(R) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} S_j^i \cdot N_{\overline{v}_i}(R \cdot |\overline{v}_i|/L_i),$$

where $N_{\overline{v}_i}(R)$ is the number of vectors of the form $\gamma\overline{v}_i$, $\gamma \in \Gamma(\omega)$, of length at most R, considered up to multiplication by ± 1. We now use the following lemma, which was proved by Veech.

Lemma 3.14 [1]. *Let $\overline{v}$ be a non-zero vector which is fixed under a non-identity element of the lattice $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$, and let $\overline{u}$ be a vector of length $|\overline{v}|^{-1}$ orthogonal to $\overline{v}$. Let α be the minimum of the quantities $\dfrac{\langle a\overline{u}, \overline{v}\rangle}{\langle \overline{v}, \overline{v}\rangle}$ over all $a \in \Gamma$, $a \neq \pm 1$,* *such that* $a\overline{v} = \pm\overline{v}$. *Then*

$$N_{\overline{v}}(R) = \big(\alpha \cdot V(\Gamma)\big)^{-1} \cdot R^2 + o(R^2) \quad as \quad R \to \infty,$$

where $V(\Gamma)$ *is the non-Euclidean area of a fundamental domain for the action of* Γ *on* $\mathbb{H}^2$.

This lemma, plus the formulae above it, give the required asymptotic behaviour for the quantities $N_0(R)$, $N(R)$, $S(R)$. Let c be the constant in the asymptotic formula for $N(R)$. Then for any $\varepsilon > 0$ and all values of R exceeding a certain $R_0 = R_0(\varepsilon)$ the following inequality holds:

$$(c - \varepsilon) \cdot R^2 \leqslant N(R) \leqslant (c + \varepsilon) \cdot R^2.$$

Since $N^*(R) = N(R) + N(R/2) + \cdots$, for any $n \in \mathbb{N}$ and $R \geqslant nR_0$ we have

$$N^*(R) \geqslant (c - \varepsilon)(1 + 2^{-2} + \cdots + n^{-2}) \cdot R^2,$$

whence

$$\liminf_{R\to\infty} N^*(R)/R^2 \geqslant c \cdot \sum_{n=1}^{\infty} n^{-2}.$$

Furthermore, for any $R \geqslant R_0$ we have

$$N^*(R) \leqslant (c + \varepsilon)(1 + 2^{-2} + \cdots + n_0^{-2}) \cdot R^2 + \sum_{n > n_0} N(R/n), \qquad n_0 = [R/R_0].$$

Since the length of a periodic trajectory is not smaller than the length of some saddle connection, $N(R) = 0$ for $R < R_1 = m(\Gamma(\omega))$. Hence,

$$\sum_{n>n_0} N(R/n) = \sum_{n_0 < n \leqslant [R/R_1]} N(R/n) \leqslant N(R_0)R/R_1.$$

So,

$$N^*(R) \leqslant (c + \varepsilon) \sum_{n=1}^{\infty} n^{-2} \cdot R^2 + R \cdot N(R_0)/R_1.$$

Since ε has been chosen arbitrarily,

$$\limsup_{R\to\infty} \frac{N^*(R)}{R^2} \leqslant c \cdot \sum_{n=1}^{\infty} n^{-2}.$$

Combining the estimates for $N^*(R)/R^2$ we obtain

$$N^*(R) = c \cdot \frac{\pi^2}{6} \cdot R^2 + o(R^2) \quad \text{as} \quad R \to \infty.$$

In a similar manner we can prove that

$$\lim_{R\to\infty} \frac{S^*(R)}{R^2} = \frac{\pi^2}{6} \cdot \lim_{R\to\infty} \frac{S(R)}{R^2}.$$

§4. Examples of Veech's theorem

4.1. Planar structures on a torus.

Proposition 4.1. *The stabilizer of a planar torus and of a planar torus with one singular point is a lattice.*

Proof. As already noted in the proof of Theorem 3.4, the stabilizer of a planar torus does not change when a singular point is added to it. Furthermore, for arbitrary planar tori ω_1 and ω_2 on surfaces M_1 and M_2, respectively, we can find a map $\varphi\colon M_1 \to M_2$ which, in the local coordinates of the atlases of ω_1 and ω_2, is an affine map with linear part a, $\det a > 0$. Here, the stabilizers $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$ are conjugate in $\mathrm{SL}(2,\mathbb{R})$: $\Gamma(\omega_2) = a \cdot \Gamma(\omega_1) \cdot a^{-1}$, so they are either both lattices or both not. Hence, it suffices to consider the planar torus $\mathbb{R}^2/\mathbb{Z}^2$. Clearly, $\Gamma(\mathbb{R}^2/\mathbb{Z}^2)$ contains the operator of rotation through the angle $\pi/2$. Moreover, since the horizontal trajectories form a single periodic pencil, of length equal to the width, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(\mathbb{R}^2/\mathbb{Z}^2)$, by Lemma 3.9. These two operators generate the *modular group* $\mathrm{SL}(2,\mathbb{Z})$, which is a lattice in $\mathrm{SL}(2,\mathbb{R})$ (see [16]). Since the stabilizer $\Gamma(\mathbb{R}^2/\mathbb{Z}^2)$ contains a lattice and is discrete (see Proposition 3.3), it is a lattice itself.

We note that for the planar torus $\mathbb{T}_{\overline{v}_1,\overline{v}_2}$, the quantity $N^*(R)$ (see §3.4) is equal to half the number of non-zero vectors in $\mathbb{Z}\overline{v}_1 \oplus \mathbb{Z}\overline{v}_2$ of length at most R. Then the results of Bleher (see [17]) imply that

$$N^*(R) = c \cdot R^2 + R^{1/2}\theta(R),$$

where c is a constant and $\theta(R)$ is a function that is almost-periodic in the sense of Besicovitch. This is a vast improvement of the estimate obtained in Theorem 3.13. It is not known whether such a representation exists for other planar structures with a lattice stabilizer.

4.2. Planar structures with one singular point. Let n and m be natural numbers, $2 \leqslant n \leqslant m$, $3 \leqslant m$. We put $k = \mathrm{LCM}(n,m)$, $N = k/n$, $M = k/m$. Let P and Q be the regular n-gon, respectively m-gon, with equal sides, with one side of P and one side of Q horizontal. If $n = 2$ we consider only the regular m-gon; if n and m are odd, we require in addition that one of P or Q lies above its horizontal side and the other below it. Let $P_1,\dots,P_N$ be the n-gons obtained from P by rotation through the angles $0, \frac{2\pi}{k},\dots,(N-1)\frac{2\pi}{k}$, and let $Q_1,\dots,Q_M$ be the m-gons obtained from Q by rotation through the angles $0, \frac{2\pi}{k},\dots,(M-1)\frac{2\pi}{k}$. We consider the disjoint union U of all $P_1,\dots,P_N,Q_1,\dots,Q_M$. In U we identify the side l_1 of P_i with the side l_2 of Q_j if l_1 and l_2 are parallel and the polygons P_i and Q_j lie on different sides (for identical orientations on l_1 and l_2). If $n = 2$, we identify sides of $Q_1,\dots,Q_M$ by this rule. By construction, each side of a polygon $P_1,\dots,P_N,Q_1,\dots,Q_M$ is identified with precisely one other side, and after the identification the union U becomes a connected, oriented, compact surface $M_{n,m}$. The identity maps on the interiors of the polygons $P_1,\dots,P_N,Q_1,\dots,Q_M$ can be uniquely extended to a planar structure $\omega_{n,m}$ on $M_{n,m}$. All vertices of the polygons are identified, giving one point in $M_{n,m}$. This point is the unique singular point heorem

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of $\omega_{n,m}$. It is removable if $n = m = 3$ or if $n = 2$ and $m = 3$, 4 or 6. The sides of the polygons $P_1, \ldots, P_N, Q_1, \ldots, Q_M$, and only they, are shortest saddle connections of $\omega_{n,m}$.

Proposition 4.2. *The planar structure corresponding to the triangle with angles* $\frac{\pi}{n}, \frac{\pi}{m}, \pi - \frac{\pi}{n} - \frac{\pi}{m}$ *(*$n, m \in \mathbb{N}$, $n \leqslant m$*) coincides with* $\omega_{n,m}$ *up to rotation and removal of certain removable singular points.*

Proof. We add to $\omega_{n,m}$ as removable singular points the centres of the regular n- and m-gons making up the surface $M_{n,m}$ (for $n = 2$, the regular 2-gon is a side of the regular m-gon). We obtain a planar structure $\widetilde{\omega}_{n,m}$. We join the centre of each regular polygon to the vertices of this polygon by line segments, and also join the centres of adjoining polygons by the mid-perpendicular to their common side. All these line segments are disjoint (except perhaps for common end-points). They are saddle connections of $\widetilde{\omega}_{n,m}$ and partition $M_{n,m}$ into equal triangles with angles $\frac{\pi}{n}$, $\frac{\pi}{m}$, $\pi - \frac{\pi}{n} - \frac{\pi}{m}$. By construction, triangles in the decomposition having a common side lie symmetrically with respect to this side, and the number of such triangles (which is equal to $2n \cdot N = 2m \cdot M = 2k$) is equal to the order of the group R generated by the linear parts of the reflections with respect to the sides of any triangle in the decomposition.

We let T be one such triangle. With each $r \in R$ we can associate the triangle T_r in the decomposition to which T is mapped by the affine transformation f_r with linear part r. We may assume that the sets of centres of the polygons $P_1, \ldots, P_N$ and of the polygons $Q_1, \ldots, Q_M$ are fixed under f_r. Thus, the triangle T_r is uniquely determined by the element r, and the correspondence $r \mapsto T_r$ is a bijection from R onto the set of triangles in the partition; if T_{r_1} and T_{r_2} are adjoining triangles, then $r_1 \circ r_2^{-1}$ is the linear part of the reflection in their common side. Then the construction in §2.2 implies that $\widetilde{\omega}_{n,m}$ is the planar structure corresponding to T.

We will now give some examples of subgroups of $\mathrm{SL}(2, \mathbb{R})$ that are lattices. Let n, m be integers, $n, m \geqslant 2$, $(n, m) \neq (2, 2)$. We put

$$L_{n,m} = \frac{\cos\frac{\pi}{n} + \cos\frac{\pi}{m}}{\sin\frac{\pi}{n}}, \qquad L_{n,\infty} = \frac{\cos\frac{\pi}{n} + 1}{\sin\frac{\pi}{n}} = \cot\frac{\pi}{2n}.$$

We denote by $\Gamma_{n,m}$ (or $\Gamma^+_{n,m}$) the subgroup of $\mathrm{SL}(2, \mathbb{R})$ generated by the elements

$$\sigma_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \quad \text{and} \quad \tau_{n,m} = \begin{pmatrix} 1 & 2L_{n,m} \\ 0 & 1 \end{pmatrix}.$$

We denote by $\Gamma^-_{n,m}$ the subgroup generated by the elements $-\sigma_n$ and $\tau_{n,m}$ ($\Gamma^-_{n,m} = \Gamma^+_{n,m}$ for n even).

Proposition 4.3. *The groups* $\Gamma^\pm_{n,m}$ $(2 \leqslant n < \infty, 2 \leqslant m \leqslant \infty, (n, m) \neq (2, 2))$ *are lattices in* $\mathrm{SL}(2, \mathbb{R})$.

Proof. We consider the Lobachevskii plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, on which the group $\mathrm{SL}(2, \mathbb{R})$ acts (see the proof of Lemma 3.7). Through the point i we draw two non-Euclidean straight lines l_1 and l_2 making angles $\frac{\pi}{n}$ with the geodesic joining i and ∞ (l_1 and l_2 are Euclidean circles with centres at the points $\cot\frac{\pi}{n}$ and $-\cot\frac{\pi}{n}$, respectively; $l_1 = l_2$ if $n = 2$). Let A_1 be the point of intersection of l_1 and the line $\operatorname{Re} z = L_{n,m}$, and let A_2 be the point of intersection of l_2 and the line $\operatorname{Re} z = -L_{n,m}$. The imaginary parts of A_1 and A_2 are equal to $\sin\frac{\pi}{m}$ for $m < \infty$, while for $m = \infty$ both A_1 and A_2 belong to the absolute $\operatorname{Im} z = 0$. The non-Euclidean 4-gon W with vertices i, A_1, ∞, A_2 has finite non-Euclidean area (see [16]). The operator σ_n (or $-\sigma_n$) maps the side iA_2 of W to the side iA_1, leaving i fixed. The operator $\tau_{n,m}$ maps the side $A_2\infty$ to the side $A_1\infty$, leaving ∞ fixed. The angle of W at i is $\frac{2\pi}{n}$, those at A_1 and A_2 are $\frac{\pi}{m}$ for $m < \infty$ and 0 for $m = \infty$. In view of the above, a theorem of Poincaré (see [16]) implies that each group $\Gamma^{\pm}_{n,m}$ is discrete and that W is a fundamental domain for its action on $\mathbb{H}^2$. Since W has finite area, $\Gamma^{\pm}_{n,m}$ is a lattice.

We note that the modular group $\mathrm{SL}(2,\mathbb{Z})$, used in the proof of Proposition 4.1, is among the listed examples of lattices; it is the group $\Gamma_{2,3}$.

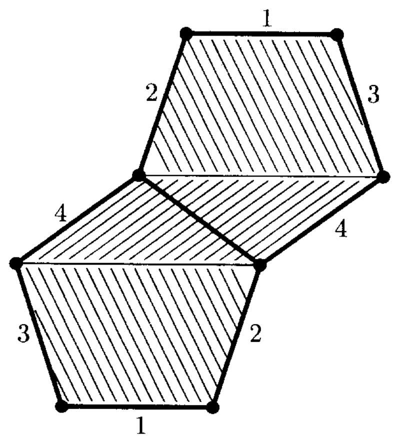

Figure 1. $\omega_{2,5} = \omega_{5,5}$

Theorem 4.4. *The stabilizers* $\Gamma(\omega_{2,n})$ *and* $\Gamma(\omega_{n,n})$ *contain the group* $\Gamma_{n,2}$ *for* n *odd and the group* $\Gamma_{n/2,\infty}$ *for* n *even. For an arbitrary* n *we have* $\Gamma_{n,3} \subset \Gamma(\omega_{n,2n})$. *Moreover,* $\Gamma_{6,\infty} \subset \Gamma(\omega_{3,4})$, $\Gamma^{-}_{15,\infty} \subset \Gamma(\omega_{3,5})$. *All stabilizers listed here are lattices.*

Proof. After rotation through an angle $\frac{2\pi}{n}$ or $\frac{2\pi}{m}$, each of the regular n- and m-gons making up $M_{n,m}$ can be made to coincide by a shift with one of the polygons listed. By the construction of $M_{n,m}$, this transformation is well defined on $M_{n,m}$ and is an affine automorphism of the planar structures $\omega_{n,m}$ and $\widetilde{\omega}_{n,m}$. Thus, the operators of rotation through $\frac{2\pi}{n}$ and $\frac{2\pi}{m}$ belong to the stabilizers $\Gamma(\omega_{n,m})$ and $\Gamma(\widetilde{\omega}_{n,m})$. In particular, we find that $\Gamma(\omega_{2,n})$ contains σ_n for n odd; $\Gamma(\omega_{2,n})$ and $\Gamma(\omega_{n,n})$ contain $\sigma_{n/2}$ for n even; $\Gamma(\omega_{n,2n})$ contains σ_n; and $\sigma_6 \in \Gamma(\omega_{3,4})$, $-\sigma_{15} \in \Gamma(\omega_{3,5})$.

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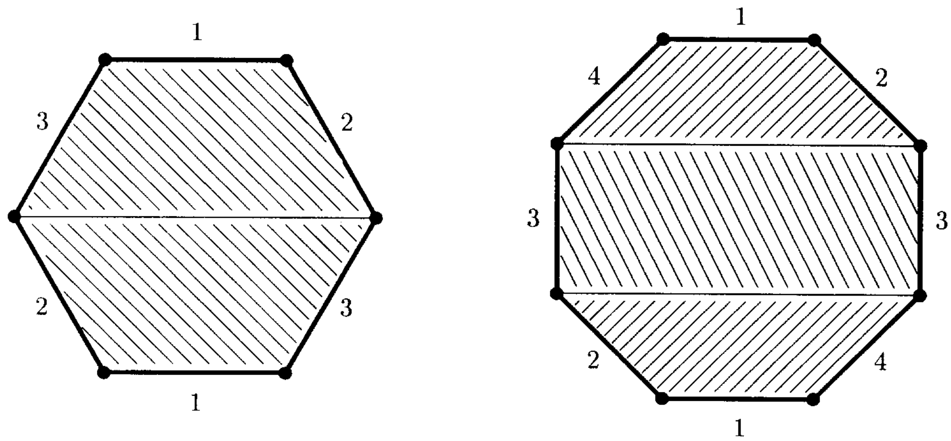


Figure 2. $\omega_{2,6}$ and $\omega_{2,8}$

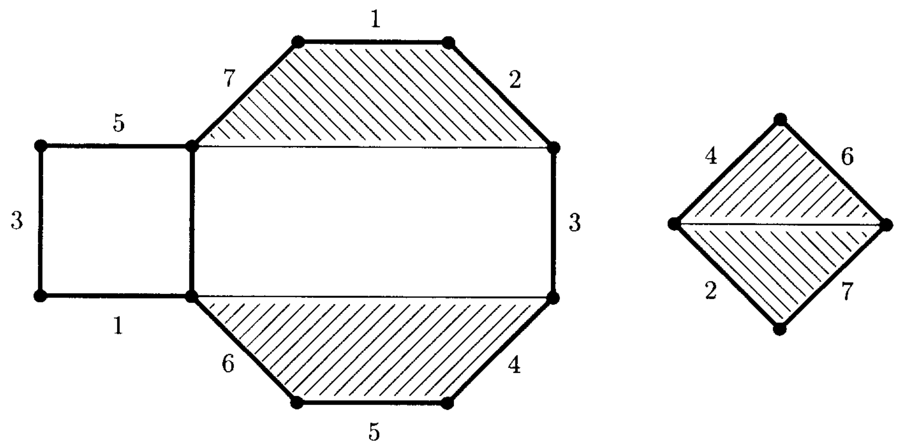


Figure 3. $\omega_{4,8}$

We now turn to the horizontal trajectories of these planar structures. The horizontal trajectories of $\omega_{n,n}$ split into $\left[\frac{n-1}{2}\right]$ pencils of periodic trajectories (see Fig. 1; here and below, identical digits indicate identified sides). The ratio of the length to the width of each pencil is $2\cot\frac{\pi}{n}$, that is, by Lemma 3.9, $\tau_{n,2} \in \Gamma(\omega_{n,n})$. Then $\Gamma_{n,2} \subset \Gamma(\omega_{n,n})$ for n odd. For n even we have $\tau_{n,2} = \tau_{n/2,\infty}$, and $\Gamma_{n/2,\infty} \subset \Gamma(\omega_{n,n})$. For n odd, $\omega_{2,n}$ coincides with $\omega_{n,n}$; for n even its horizontal trajectories form $[n/4]$ periodic pencils. If n is not divisible by 4, then the ratio of the length to the width of each pencil is $2\cot\frac{\pi}{n}$. If n is divisible by 4, all pencils except one have this ratio (see Fig. 2). For this pencil, which contains the centre of the regular n-gon (from which $M_{2,n}$ is made up), the ratio of the length to the width is $\cot\frac{\pi}{n}$. In any case, $\Gamma(\omega_{2,n})$ contains $\tau_{n/2,\infty}$, and so $\Gamma_{n/2,\infty} \subset \Gamma(\omega_{2,n})$. Furthermore, the horizontal trajectories of $\omega_{n,2n}$ form $n-1$ periodic pencils (see Fig. 3), the ratio of the length to the width for each of these being

$$\cot\frac{\pi}{2n} + \cot\frac{\pi}{n} = \frac{\cot\frac{\pi}{2n}\sin\frac{\pi}{n} + \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} = \frac{(1+\cos\frac{\pi}{n}) + \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} = 2L_{n,3}.$$

By Lemma 3.9 we have $\tau_{n,3} \in \Gamma(\omega_{n,2n})$, and $\Gamma(\omega_{n,2n}) \supset \Gamma_{n,3}$.

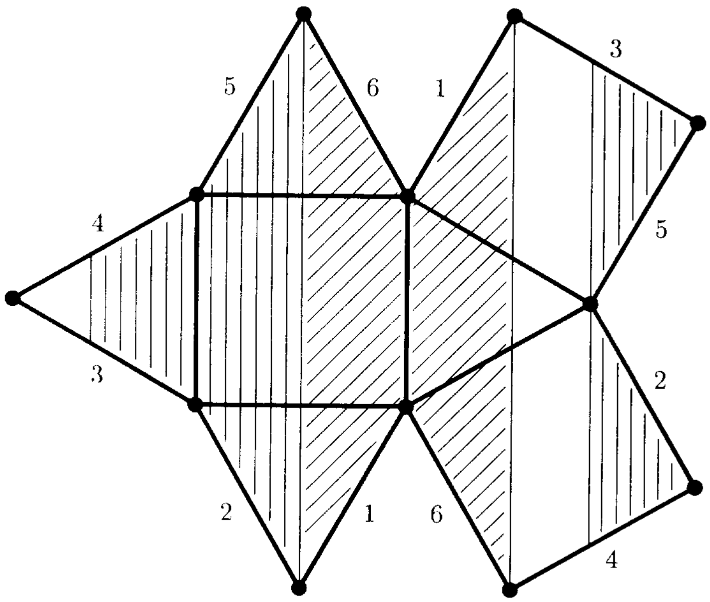
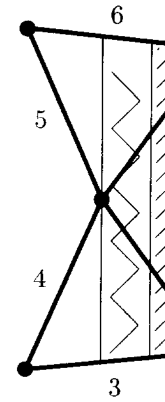


Figure 4. $\omega_{3,4}$, rotated through 90°

The horizontal trajectories of $\omega_{3,4}$ split into 3 periodic pencils (see Fig. 4). The two pencils bounded by shortest saddle connections (hatched in the figure) have length $2d(1+\cos\frac{\pi}{6})$ and width $d\sin\frac{\pi}{6}$, where d is the length of the shortest saddle connection. So, the ratio of length to width is equal to $2L_{6,\infty}$. The third pencil is readily seen to have this ratio equal to $2\cot\frac{\pi}{12}$. Consequently, $\tau_{6,\infty} \in \Gamma(\omega_{3,4})$ and $\Gamma(\omega_{3,4}) \supset \Gamma_{6,\infty}$.

The horizontal trajectories of $\omega_{3,5}$ split into 4 periodic pencils (see Fig. 5). The length and width of the pencil indicated by straight hatching in the figure are equal to $2d(1+\cos\frac{\pi}{15})$ and $d\sin\frac{\pi}{15}$, where d is the length of the shortest saddle connection. The length and width of the pencil indicated by skew hatching in the figure are equal to $2D(1+\cos\frac{\pi}{15})$ and $D\sin\frac{\pi}{15}$, where D is the length of the diagonal of the regular 5-gon with side d. In both cases the ratio of length to width is equal to $2L_{15,\infty}$. The ratio of length to width for the pencil indicated by the broken lines in the figure is equal to $2\cot\frac{\pi}{30} = 2L_{15,\infty}$. Finally, the pencil not marked at all has length $2d(1+2\cos\frac{\pi}{15}+\cos\frac{2\pi}{15})$ and width $d\sin(\pi/15)$. Their ratio is

$$2\,\frac{1+2\cos\frac{\pi}{15}+\cos\frac{2\pi}{15}}{\sin\frac{2\pi}{15}} = 2\,\frac{2\cos^2\frac{\pi}{15}+2\cos\frac{\pi}{15}}{2\sin\frac{\pi}{15}\cos\frac{\pi}{15}} = 2\,\frac{\cos\frac{\pi}{15}+1}{\sin\frac{\pi}{15}} = 2L_{15,\infty}.$$



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4.3. Non-lattice stabi
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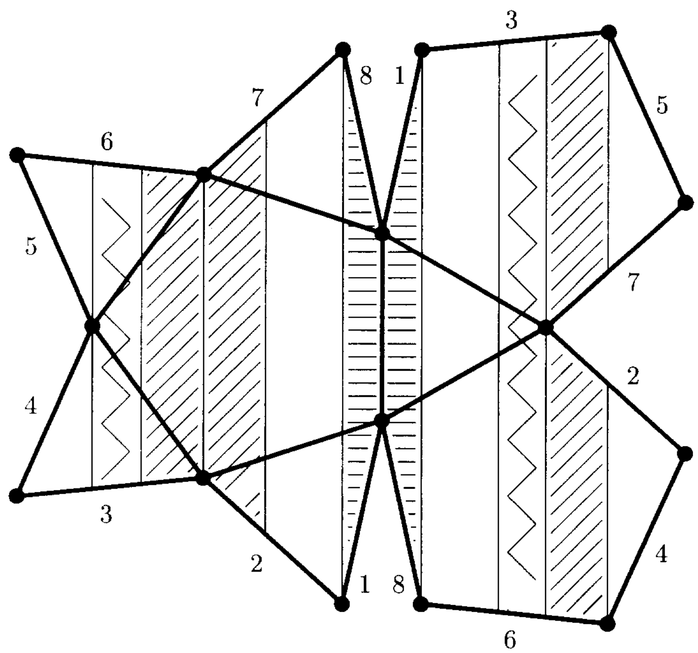

Figure 5. $\omega_{3,5}$, rotated through 90°

So, by Lemma 3.9, $\tau_{15,\infty} \in \Gamma(\omega_{3,5})$ and $\Gamma(\omega_{3,5}) \supset \Gamma^-_{15,\infty}$. Thus, the stabilizer of each of these planar structures contains a lattice. Hence, in view of the fact that the stabilizers are discrete, they themselves must be lattices.

We conclude by noting that in [1] the presence of a lattice stabilizer was proved for the planar structures $\omega_{2,n}$ and $\omega_{n,n}$ ($n \geqslant 3$), was announced for $\omega_{3,4}$, and was conjectured for $\omega_{n,2n}$ ($n \geqslant 2$).

4.3. Non-lattice stabilizers. Theorem 3.4 (together with Lemma 3.8) makes it possible to find examples of planar structures whose stabilizers are not lattices.

Theorem 4.5. *The stabilizers of* $\omega_{4,6}$ *and* $\omega_{4,12}$ *are not lattices in* $\mathrm{SL}(2,\mathbb{R})$.

Proof. The horizontal trajectories of $\omega_{4,6}$ split into 4 pencils of periodic trajectories (see Fig. 6). The ratio of the length to the width for the pencil indicated by skew hatching in the figure is equal to $r_1 = 1 + \cot\frac{\pi}{6} = 1 + \sqrt{3}$. On the other hand, the pencil not marked in the figure has length $2d(1 + \sqrt{2}\cos\frac{\pi}{12}) = d(3 + \sqrt{3})$ and width $d\sqrt{2}\sin\frac{\pi}{12} = d(\sqrt{3} - 1)/2$ (as in the proof of Theorem 4.4, d is the length of the shortest saddle connection). Their ratio is $r_2 = 6 + 4\sqrt{3}$. The numbers r_1 and r_2 are incommensurate, hence, by Theorem 3.4 and Lemma 3.8, the stabilizer $\Gamma(\omega_{4,6})$ is not a lattice.

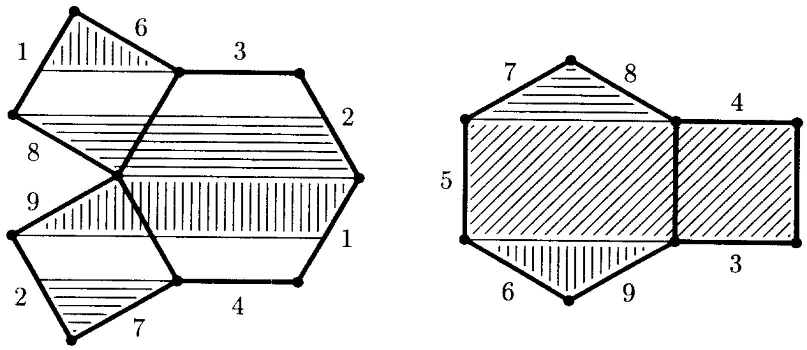

Figure 6. $\omega_{4,6}$

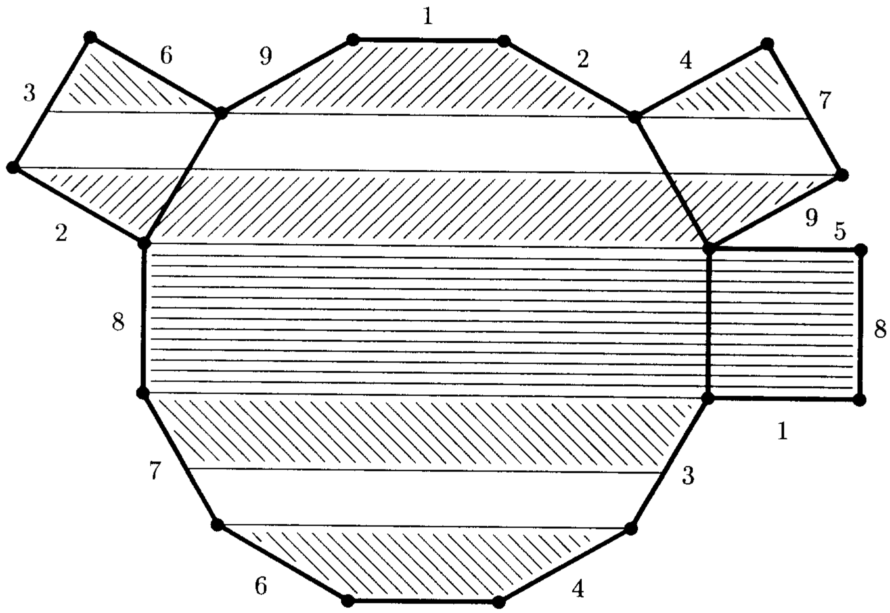

Figure 7. $\omega_{4,12}$

One can similarly treat the case $\omega_{4,12}$. Its horizontal trajectories split into 4 periodic pencils (see Fig. 7). The ratio of the length to the width for the pencil marked by straight hatching in the figure is equal to $r_1 = 1 + \cot\frac{\pi}{12} = 3 + \sqrt{3}$. The pencil marked by skew hatching in the figure has length $2d(1 + 2\cos\frac{\pi}{6} + \cos\frac{\pi}{3})$ and width $d\sin\frac{\pi}{6}$; their ratio is $r_2 = 6 + 4\sqrt{3}$. The numbers r_1 and r_2 are incommensurate again, hence the stabilizer $\Gamma(\omega_{4,12})$ is not a lattice.

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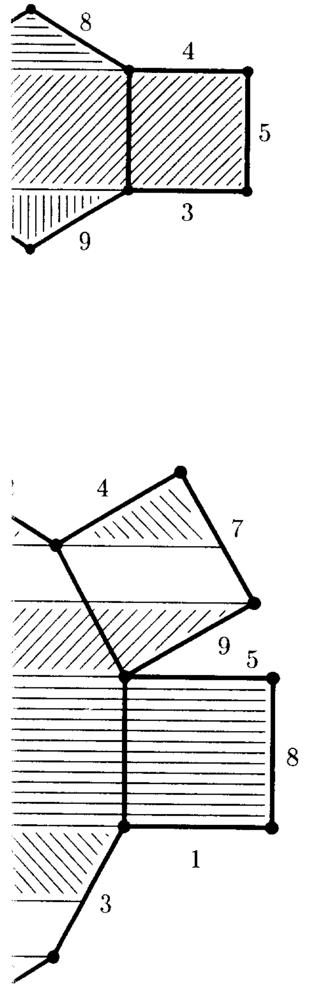

§5. Covers of planar structures

5.1. Finiteness of the number of covers.

Definition 5.1. Let (M_1,ω_1) and (M_2,ω_2) be surfaces with planar structures. ω_1 is said to *cover* ω_2 if there is a continuous map $f\colon M_1 \to M_2$ (called a *cover*) mapping the singular points of ω_1 to the singular points of ω_2 and acting as a shift in the local coordinates given by the atlases of ω_1 and ω_2.

In view of the compactness of the surfaces under consideration, the number of preimages of a point $x \in M_2$ under f is finite. In view of the connectedness of the surfaces, this number is the same for all non-singular points. It is called the *multiplicity* of the cover f. If $x \in M_2$ is a singular point of multiplicity m and $x_1,\dots,x_k \in M_1$ are its preimages with respective multiplicities $m_1,\dots,m_k$, then m divides each m_i and $\sum_{i=1}^k m_i = mN$, where N is the multiplicity of the cover.

The following construction gives examples of covers of planar structures. Suppose we partition a rational polygon Q into equal polygons $P = P_1, P_2, \dots, P_n$, any two of these intersecting only along common sides and at common vertices, and polygons with a common side lying symmetrically with respect to this side. Moreover, we assign to each P_i a $+$ or $-$ sign in such a way that polygons with a common side have different signs (if P is not axially symmetric, this condition follows from the previous ones). The construction in §2.2 associates with P and Q planar structures ω_P and ω_Q on certain surfaces M_P and M_Q.

Proposition 5.1. *The planar structure ω_Q covers ω_P, possibly after the addition to ω_Q or removal from ω_P of a certain number of removable singular points. The multiplicity of the cover is a divisor of n.*

Proof. From the construction it follows that there are affine maps $\alpha_1,\dots,\alpha_n$ such that α_i maps P_i to P and, for arbitrary P_i and P_j with a common side, $\alpha_j^{-1}\alpha_i$ is the symmetry with respect to this side. The maps $\alpha_1,\dots,\alpha_n$ can be extended to a single continuous map $f\colon Q \to P$.

Let $R(Q), R(P_1),\dots,R(P_n)$ be the groups generated by the linear parts of the reflections in the sides of the polygons. By construction, $R(P_1) = \dots = R(P_n)$, so that $R(Q) \subset R(P)$. We define the map $g\colon Q \times R(Q) \to P \times R(P)$ by $g(x,r) = (f(x), r\widetilde{\alpha}_i^{-1})$, where i is such that $x \in P_i$ and $\widetilde{\alpha}_i$ is the linear part of α_i (clearly, $\widetilde{\alpha}_i \in R(P)$). Then the spaces $Q \times R(Q)$ and $P \times R(P)$ can be factored to surfaces M_Q and M_P and the map g can be factored to a well-defined map $h\colon M_Q \to M_P$. Now, h is a cover of the planar structures ω_Q and ω_P of multiplicity n/m, where m is the index of the subgroup $R(Q)$ in $R(P)$, except that certain non-singular points on M_Q can be mapped to (removable) singular points on M_P. To be precise, these are points of the form (x,r), where x is a vertex of a polygon P_i that is not a vertex of Q. To correct this situation it suffices to add such points to the singular points of ω_Q. If none of the points in the preimage $f^{-1}(f(x))$ is a vertex of Q, instead of this we may regard the points $(f(x), r) \in M_P$ as singular for ω_P.

Definition 5.2. Let (M,ω), (M_1,ω_1) and (M_2,ω_2) be surfaces with planar structures. Two covers of planar structures $f_1\colon M_1 \to M$ and $f_2\colon M_2 \to M$ are called *isomorphic* if there is an isomorphism $\varphi\colon M_1 \to M_2$ of ω_1 and ω_2 such that $f_1 = f_2 \circ \varphi$. We can similarly define the isomorphism of covers $g_1\colon M \to M_1$ and $g_2\colon M \to M_2$.

Proposition 5.2. *A planar structure ω on a surface M admits only a finite number of covers of given multiplicity, up to isomorphism. This number is bounded by a constant depending on the genus of M, the number of removable singularities of ω, and the multiplicity of the cover.*

Proof. We consider an arbitrary triangulation of M for which the vertices are the singular points of ω, the edges are the saddle connections, and the faces do not contain interior singular points. The proof of Theorem 2.9 implies that the number of edges of the triangulation is $3(k-\chi)$, where k is the number of singular points of ω and χ is the Euler characteristic of M. We choose edges (saddle connections) $L_1, \dots, L_n$ such that the domain M_0 obtained from M by deleting the singular points and the selected saddle connections becomes connected and simply-connected. We cut M along each L_i, and we denote the boundaries of the cut by L_i^+ and L_i^-. By adding to M_0 the boundaries of the cuts we obtain a compact simply-connected surface M' with boundary. We now take N copies of M' (that is, the direct product $M' \times \{1,2,\dots,N\}$), given by permutations $\pi_1, \dots, \pi_n$ on $\{1,2,\dots,N\}$, and identify the segments $L_i^+ \times \{j\}$ and $L_i^- \times \{\pi_i(j)\}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant N$. We obtain a compact surface $M'' = M''(\pi_1, \dots, \pi_n)$ without boundary. The surface M'' is connected if the group generated by the permutations $\pi_1, \dots, \pi_n$ acts transitively on $\{1,2,\dots,N\}$. The planar structure on $M_0 \times \{1,2,\dots,N\} \subset M''$ induced by restricting ω to M_0 can be uniquely extended to a planar structure $\omega(\pi_1, \dots, \pi_n)$ on M''. Moreover, the natural projection $p\colon M'' \to M$ is an N-fold cover of the planar structures $\omega(\pi_1, \dots, \pi_n)$ and ω. The total number of covers thus constructed does not exceed $(N!)^n$. By construction, $n \leqslant 3(k-\chi)$. By Theorem 2.9, the number of non-removable singular points of ω is at most $-\chi$. Hence, $(N!)^n$ is bounded by a constant depending on χ, N and the number of removable singular points of ω.

To finish the proof it now suffices to show that every N-fold cover f of ω by some planar structure ω_1 on a surface M_1 is isomorphic to one of the covers constructed above. We take a point $x \in M_0$ and let $x_1, \dots, x_N$ be its preimages under f. Since M_0 is simply-connected, there are unique maps $f_1, \dots, f_N$ from M_0 into M_1 that are right-inverse to f and such that $f_i(x) = x_i$, $1 \leqslant i \leqslant N$. Moreover, the domains $f_1(M_0), \dots, f_N(M_0)$ are pairwise disjoint and are separated from each other by the preimages of the saddle connections $L_1, \dots, L_n$ (each saddle connection has N preimages). This implies that the map $\varphi\colon M_0 \times \{1,2,\dots,N\} \to M_1$ given by $\varphi(y,i) = f_i(y)$ can, for a certain choice of permutations $\pi_1, \dots, \pi_n$, be extended to a continuous map $\varphi\colon M''(\pi_1, \dots, \pi_n) \to M_1$. By construction, $f \circ \varphi = p$ and φ is an isomorphism of the planar structures $\omega(\pi_1, \dots, \pi_n)$ and ω_1. Thus, the cover f is isomorphic to the cover p.

Proposition 5.3. *The number of covers realized by a planar structure ω is finite, up to isomorphism, and bounded by a constant depending on the genus of M and the number of singular points of ω.*

Proof. Let (M_1, ω_1) and (M_2, ω_2) be surfaces with planar structures, and let $f_1 \colon M \to M_1$ and $f_2 \colon M \to M_2$ be covers of the planar structures. Let S be the set of points $x \in M_1$ such that the preimages $f_1^{-1}(x)$ are mapped by f_2 to a single point of M_2. The definition of cover implies that S is closed and that each non-singular point of it lies in its interior. Thus, $S = M_1$ so long as S contains at least one non-singular point. Moreover, we can define a map $\varphi \colon M_1 \to M_2$ such that $f_2 = \varphi \circ f_1$. Clearly, φ is a cover of the planar structures ω_1 and ω_2. If f_1 and f_2 have the same multiplicity, then φ is an isomorphism, that is, f_1 and f_2 are isomorphic covers.

We now choose an arbitrary direction $\overline{v}$ and a small ε such that the ε-neighbourhoods of the singular points of ω do not intersect. From the singular points of ω we draw the geodesic intervals in the direction $\overline{v}$ of length ε. Let $x_1, \dots, x_n$ be the points at which these intervals end (there are as many as the sum of the multiplicities of the singular points). For each cover $f \colon M \to \widetilde{M}$ we let $S(f)$ be the preimage of $f^{-1}(f(x_1))$. Clearly, $S(f) \subset \{x_1, \dots, x_n\}$. From what was said above it follows that the covers f_1 and f_2 realized by ω are isomorphic if $S(f_1) = S(f_2)$. Thus, up to isomorphism there are at most as many distinct covers as there are subsets of $\{x_2, \dots, x_n\}$, that is, 2^{n-1}. This finishes the proof, since, by Theorem 2.9, $n = k - \chi$, where k is the number of singular points of ω and χ is the Euler characteristic of M.

We note that the proof of Proposition 5.2 includes a construction of all (up to isomorphism) covers of a planar structure ω. A similar construction for covers realizable by a planar structure ω cannot be found in the proof of Proposition 5.3.

5.2. Relation with stabilizers.

Theorem 5.4. *Suppose that a planar structure ω_1 is covered by a planar structure ω_2. Then the group $\Gamma(\omega_1) \cap \Gamma(\omega_2)$ is a subgroup of finite index in each of the groups $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$. In particular, $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$ are both lattices or are both not.*

Proof. Let M_1 be the surface on which ω_1 is given. Let S be the set of triples (M, ω, f) with ω a planar structure on the surface M and $f \colon M_1 \to M$ a cover of ω by ω_1, and let $\widetilde{S}$ be the set of such triples regarded up to isomorphism of covers. The map f realizes also a cover of the planar structures $a\omega_1$ and $a\omega$, with $a \in \mathrm{GL}(2, \mathbb{R})$. Furthermore, an arbitrary element φ of the group G of affine automorphisms of ω_1 is an isomorphism of the planar structures ω_1 and $a(\varphi)^{-1}\omega_1$, where $a(\varphi)$ is the linear part of φ. Thus, a right action of G is defined on S: $(M, \omega, f)\varphi = (M, a(\varphi)^{-1}\omega, f \circ \varphi)$. We note that isomorphic covers are mapped to isomorphic covers, that is, the action factorizes to an action of G on $\widetilde{S}$. Since by Proposition 5.2 the set $\widetilde{S}$ is finite, there is a subgroup $G_0 \subset G$ of finite index that acts as the identity on $\widetilde{S}$. Let Γ be the group of linear parts of the affine automorphisms in G_0 that preserve the orientation on M_1. Clearly, Γ is a subgroup of finite index in $\Gamma(\omega_1)$. On the other hand, since G_0 acts as the identity on $\widetilde{S}$, it follows that Γ is contained in the stabilizer of every planar structure covered by ω_1. In particular, $\Gamma \subset \Gamma(\omega_1) \cap \Gamma(\omega_2)$, that is, $\Gamma(\omega_1) \cap \Gamma(\omega_2)$ is a subgroup of finite index in $\Gamma(\omega_1)$.

The finiteness of the index of $\Gamma(\omega_1) \cap \Gamma(\omega_2)$ in $\Gamma(\omega_2)$ can be similarly derived from Proposition 5.3.

Proposition 5.5. *The stabilizer of a planar structure corresponding to a regular n-gon is a lattice.*

Proof. By joining the centre of the regular n-gon to the vertices and the mid-points of the sides using straight line segments, we obtain a decomposition of the regular n-gon into triangles with angles $\pi/2$ and π/n. Proposition 5.1 implies that the planar structure ω_n corresponding to the regular n-gon covers the planar structure $\widetilde{\omega}_n$ corresponding to a triangle in the decomposition; moreover, in the latter we have to take into account the non-singular points arising from the vertices with angles $\pi/2$ and π/n. With this understanding, $\Gamma(\widetilde{\omega}_n)$ is a lattice, as has been shown in §4.2. By Theorem 5.4, $\Gamma(\omega_n)$ is also a lattice.

Proposition 5.5 was proved by Veech in [2], where he also studied the planar structures ω_n and their stabilizers in some detail.

The following example shows that we have to consider removable singularities. An isosceles triangle with angle $2\pi/n$ at the vertex is partitioned by the altitude into two triangles with angles $\pi/2$ and π/n. Therefore the planar structure ω_n' corresponding to it covers $\widetilde{\omega}_n$, where in this case we have to take as non-singular the points corresponding to the angle $\pi/2$ but not those corresponding to π/n. We can show that with this understanding, $\Gamma(\widetilde{\omega}_n)$ is not a lattice for any odd $n > 3$; consequently, $\Gamma(\omega_n')$ is also not a lattice. In particular, it is not known whether ω_n' is an elementary planar structure for these values of n.

We give a sufficient condition for $\Gamma(\omega) \subset \Gamma(\widetilde{\omega})$, given that ω covers $\widetilde{\omega}$.

Proposition 5.6. *Let f be a cover of a planar structure $\widetilde{\omega}$ by a planar structure ω, having multiplicity N. If all covers of multiplicity N realized by ω are pairwise isomorphic, then $\Gamma(\omega) \subset \Gamma(\widetilde{\omega})$. In particular, this is true if for some $\overline{v} \in \mathbb{R}^2$ the planar structure $\widetilde{\omega}$ has a unique saddle connection with development $\overline{v}$; for example, $\Gamma(\omega_n) \subset \Gamma(\widetilde{\omega}_n)$.*

Proof. We assume that all covers of multiplicity N realized by ω are isomorphic to f. We choose an element $a \in \Gamma(\omega)$ and let φ be an affine automorphism of ω with linear part a^{-1}. Then $f \circ \varphi$ is a cover of the planar structures ω and $a\widetilde{\omega}$. The multiplicity of $f \circ \varphi$ is N, hence f and $f \circ \varphi$ are isomorphic. This implies that $\widetilde{\omega}$ and $a\widetilde{\omega}$ are isomorphic, that is, $a \in \Gamma(\widetilde{\omega})$. Since a is arbitrary, this implies that $\Gamma(\omega) \subset \Gamma(\widetilde{\omega})$.

Now we assume that $\widetilde{\omega}$ has a unique saddle connection with development $\overline{v}$. Then ω has exactly N saddle connections with this development. Clearly, for any cover of multiplicity N these saddle connections become a single saddle connection. Therefore, the proof of Proposition 5.3 implies that all such covers are isomorphic.

We finally show that $\Gamma(\omega_n) \subset \Gamma(\widetilde{\omega}_n)$. In fact, ω_n covers $\widetilde{\omega}_n$, and all shortest saddle connections of $\widetilde{\omega}_n$ (their preimages under the cover correspond to the sides of the regular n-gon) have distinct developments.

§6. Another proof of Veech's theorem

6.1. Property A.

Definition 6.1. Let ω be a planar structure on a surface M. An *ω-triangle* is a triangle on M whose vertices are singular points, whose sides are saddle connections, and whose interior does not contain singular points.

Definition 6.2. A planar structure ω is said to have *property* A if the area of each ω-triangle is larger than a positive constant. ω is said to have *property* B if, moreover, the area of each ω-triangle can only assume finitely many values.

Proposition 6.1. *Property* A *is equivalent to the following: if a planar structure ω has a saddle connection in some direction, then all trajectories in that direction are either periodic or saddle connections; moreover, the ratio of the lengths of parallel saddle connections is bounded above by a constant which does not depend on the direction.*

Proof. We assume that ω has property A, and let δ be a positive constant smaller than the area of every ω-triangle. We consider an arbitrary saddle connection L. Through a point $x \in L$ we draw the geodesic interval J of length $\varepsilon = 2\delta/l$, where l is the length of L, perpendicular to L. The trajectories emitted from any point of this interval distinct from x and parallel to L do not hit a singular point (for otherwise there would be an ω-triangle with L as one of its sides and with altitude perpendicular to L not exceeding J; the area of this ω-triangle would be no larger than $1/2 \cdot l \cdot \varepsilon = \delta$). This implies that all such trajectories are periodic. Thus, an arbitrary saddle connection bounds 2 (possibly coincident) pencils of periodic trajectories. On the other hand, there are only finitely many pencils of periodic trajectories parallel to the given direction, and each such pencil is bounded by saddle connections. Therefore the saddle connections and periodic pencils parallel to L cover the whole surface M, since M is connected.

Furthermore, if a saddle connection L bounds a pencil of periodic trajectories of width ε, then there is an ω-triangle with L as one of its sides and whose altitude perpendicular to L is equal to ε. This implies that the ratio of the lengths of two saddle connections bounding the same pencil is equal to the ratio of the areas of certain ω-triangles, and therefore does not exceed S/δ, where S is the area of M. In view of the assertions proved above, for any pair of parallel saddle connections L and L' we can find a chain of pairwise distinct saddle connections $L_0 = L, L_1, \dots, L_k = L'$ in which any pair of adjacent saddle connections bounds a single pencil. The number of saddle connections and the number of periodic pencils parallel to the given direction do not exceed a certain number k_0 which depends on ω only. Consequently, the ratio of the lengths of parallel saddle connections does not exceed the constant $(S/\delta)^{k_0-1}$, as required.

We now turn to the second part of the assertion. We assume that ω is such that the geodesic flow in the direction parallel to the saddle connection splits completely into pencils of periodic trajectories and, moreover, we assume that the ratio of the lengths of parallel saddle connections is bounded above by a constant C. Further, let $\overline{v}$ be the direction parallel to the saddle connection, and P an arbitrary pencil of periodic trajectories in this direction. We choose the saddle connection L that intersects the trajectories of P and does not leave its boundaries (the end-points of L lie on oppositely located sides of the pencil). Let l be the length of L, and let α be the angle between L and the trajectories in the pencil. Then the width of P is $l \sin\alpha$. Furthermore, let P_1 be another pencil of periodic trajectories in the direction $\overline{v}$. Since all trajectories parallel to L_1 are either periodic or saddle connections, we can find a saddle connection L_1 parallel to L and intersecting the trajectories of P_1 (in general, L_1 goes beyond the boundary of P_1). If l_1 denotes the length of L_1, then the width of P_1 is at most $l_1 \sin\alpha$, so that the ratio of the widths of P_1 and P is at most $\frac{l_1 \sin\alpha}{l \sin\alpha} = \frac{l_1}{l} \leqslant C$. The ratio of their lengths does not exceed Ck_1, where k_1 is the number of saddle connections in the direction $\overline{v}$. Consequently, the ratio of the areas of P_1 and P is at most $C^2 k_1$. We denote by k_2 the number of pencils of periodic trajectories parallel to $\overline{v}$. The total sum of the areas of all pencils is S, the area of M, so that the area of each pencil is at least $\frac{S}{1+C^2k_1(k_2-1)}$. As already noted above, k_1 and k_2 are bounded above by the constant k_0, which does not depend on the direction $\overline{v}$. Consequently, the area of an arbitrary pencil of periodic trajectories is at least equal to the constant $\frac{S}{1+C^2k_0(k_0-1)}$.

We now consider an arbitrary ω-triangle T. Let L be a side of it and let P be the pencil of periodic trajectories bounded by the saddle connection L and lying on the same side of L as T. Clearly, the width of P is not larger than the altitude of T perpendicular to L, and its length is not larger than Ck_0 times the length of L. So, the ratio of the area of the pencil to the area of the ω-triangle is at most $2Ck_0$, and hence the area of the ω-triangle is at least equal to the constant $\frac{S}{1+C^2k_0(k_0-1)} \cdot \frac{1}{2Ck_0} > 0$. Thus, the ω-triangle has the property A.

Proposition 6.2. *Suppose that the planar structure ω has property* A. *Then the area of each pencil of periodic trajectories is larger than a positive constant. If $r_1, r_2, \dots, r_k$ are the ratios of the length to the width for all pencils of periodic trajectories in a certain direction (parallel to a saddle connection), then $r_1, r_2, \dots, r_k$ are rationally commensurate and the ratio $\dfrac{\mathrm{LCM}(r_1, r_2, \dots, r_k)}{r_i}$ is bounded above by a constant which does not depend on the direction.* (See §3.2 for the definition of $\mathrm{LCM}(r_1, r_2, \dots, r_k)$.)

Proof. The fact that the area of each pencil of periodic trajectories is larger than a positive constant S_0 is obvious, since each pencil contains an ω-triangle. Let P_1 and P_2 be two pencils of periodic trajectories bounded by a single saddle connection L of length l, and let r_1 and r_2 be the ratios of the length to the width for each of them. We will show that r_1 and r_2 are commensurate and that their least common multiple exceeds them by at most $k_0 S/S_0$ times, where S is the area of M and k_0 is the maximum possible number of distinct saddle connections or periodic pencils in a single direction.

Without loss of generality we may assume that P_1 and P_2 are horizontal. Let L_0 be the saddle connection intersecting the trajectories of P_1, not going beyond the boundaries of P_1 and having common end-point O with L. We put

$$h_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{for all} \quad x \in \mathbb{R}.$$

Clearly, each ω-triangle is also an $h_x\omega$-triangle, with the same area. For some $a \in \mathbb{R}$ the saddle connection L_0 is vertical with respect to $h_a\omega$. In this case we can find for any $n \in \mathbb{Z}$ a saddle connection L_n intersecting P_1 that goes out from O and is vertical with respect to $h_{a+nr_1}\omega$. Moreover, L and L_n are legs of the right-angled $h_{a+nr_1}\omega$-triangle T_n. Let Q be the singular point on the side of P_2 lying opposite L, and let Q_n be the orthogonal (with respect to $h_{a+nr_1}\omega$) projection of Q on the side of the pencil containing L. If for some $n \in \mathbb{Z}$ the point Q_n lies on L, then there is clearly an $h_{a+nr_1}\omega$-triangle T with side L_n such that Q is the vertex opposite to L_n, while the altitude from Q is equal to the interval Q_nO. In this case the ratio of the length of Q_nO to l is equal to the ratio of the areas of T and T_n, and hence it is larger than S_0/S.

The distance between points Q_n and Q_{n+1} on the side of P_2, divided by the length l_2 of the pencil, is, up to an integer factor, equal to r_1/r_2. This immediately implies that r_1 and r_2 are commensurate, since otherwise the points Q_n would be everywhere dense on L. Let r be the least common multiple of r_1 and r_2. The points Q_n divide the side of P_2 into r/r_1 equal parts. Now, if $l' = l_2r_1/r$ does not exceed the length l of L, then there exists an $n \in \mathbb{Z}$ such that Q_n belongs to L and the length of Q_nO does not exceed l'. In view of what was said above, $l'/l > S_0/S$. Since $l_2/l \leqslant k_0$, we arrive at the required estimate $r/r_1 < k_0S/S_0$. The similar estimate for r/r_2 can be obtained by interchanging P_1 and P_2 in the above reasoning.

All pencils of periodic trajectories in a single direction can be arranged in a sequence $P_1, P_2, \dots, P_k$, where each pencil, from P_2 onwards, is adjacent (that is, has a common saddle connection on its boundary) to some previous one. Let $r_1, \dots, r_k$ be the ratios of the length to the width of $P_1, \dots, P_k$. Using what was proved above, by induction with respect to i we find that the least common multiple of the numbers $r_1, \dots, r_i$ $(1 \leqslant i \leqslant k)$ exceeds each of these numbers by at most a factor C^{i-1}, $C = k_0S/S_0$. Since $k \leqslant k_0$, we have $\dfrac{\mathrm{LCM}(r_1, \dots, r_k)}{r_i} \leqslant C^{k_0-1}$, where the constant C^{k_0-1} is independent of the direction.

Taking into account Lemma 3.9, Propositions 6.1 and 6.2 imply that for any vector $\overline{v}$ parallel to a saddle connection, the stabilizer $\Gamma(\omega)$ contains an element a such that $a\overline{v} = \overline{v}$. Thus, $\Gamma(\omega)$ is a sufficiently rich group.

Proposition 6.3. *Suppose that the planar structure ω has property* A. *Let $\overline{v}$ be a vector not parallel to any saddle connection. Then the geodesic flow on M in the direction $\overline{v}$ is strongly ergodic.*

Proof. We prove this by contradiction. We assume that the geodesic flow on M in the direction $\overline{v}$ is not strongly ergodic. Since $\overline{v}$ is not parallel to a saddle connection, this flow must be minimal, hence we can use Theorem 3.11. So, for $\varepsilon > 0$ we can find an $l_0 = l_0(\varepsilon)$ such that for all $l \geqslant l_0$ there is a saddle connection $L_{l,\varepsilon}$ of length at most l and whose projection on the direction $\overline{u}$ perpendicular to $\overline{v}$ has length less than ε/l. Let L' be the shortest saddle connection that we can take for $L_{l_0,\varepsilon}$, and let h be its projection on the direction $\overline{u}$ ($h < \varepsilon/l_0$). Let L'' be the saddle connection $L_{l_1,\varepsilon}$, where $l_1 = \varepsilon/h$. Its projection on the direction $\overline{u}$ is smaller than $\varepsilon/l_1 = h$, so, by the choice of L', the saddle connection L'' is not shorter than L'. This implies that they are not parallel.

Let l', l'' be the lengths of L', L'', let α', α'' be the angles they form with the direction $\overline{v}$, and let α be the angle between the saddle connections themselves ($\alpha, \alpha', \alpha'' \in [0, \pi/2]$). We have: $l_1 \geqslant l'' \geqslant l'$, $0 < \alpha \leqslant \alpha' + \alpha''$. By construction, $\sin\alpha' = h/l'$, $\sin\alpha'' < h/l''$. On the other hand, the projection of L'' on the direction perpendicular to L' is at least w' (the width of some pencil of periodic trajectories parallel to L'). The proof of Proposition 6.1 implies that $w' \geqslant C/l'$, where C is a constant depending on the planar structure only. As a result we find that $\sin\alpha \geqslant \frac{w'}{l''} \geqslant \frac{C}{l' \cdot l''}$. Since $\frac{2}{\pi}\beta \leqslant \sin\beta \leqslant \beta$ for $0 \leqslant \beta \leqslant \pi/2$, we obtain:

$$\frac{C}{l' \cdot l''} \leqslant \sin\alpha \leqslant \alpha \leqslant \alpha' + \alpha'' \leqslant \frac{2}{\pi}(\sin\alpha' + \sin\alpha'') < \frac{2h}{\pi}\Big(\frac{1}{l'} + \frac{1}{l''}\Big),$$

whence

$$C < \frac{2h}{\pi}(l' + l'') = \frac{2}{\pi}\frac{\varepsilon}{l_1}(l' + l'') \leqslant \frac{4}{\pi}\varepsilon.$$

Thus, $\varepsilon > \frac{\pi}{4}C$, that is, ε cannot be made arbitrarily small, contradicting the assumption. This proves the proposition.

We summarize the results obtained in this subsection.

Proposition 6.4. *A planar structure having property* A *satisfies the Veech alternative.*

So, from the point of view of the behaviour of geodesic flows, planar structures having property A do not differ from planar structures having a lattice stabilizer. A difference between them will be established in the next subsection.

6.2. Property B.

We will first prove certain assertions concerning lattices in $\mathrm{SL}(2,\mathbb{R})$ that will be used in the proof of Theorem 6.8.

Definition 6.3. A non-zero vector $\overline{v} \in \mathbb{R}^2$ is called a *parabolic vector* of a discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ if $a\overline{v} = \overline{v}$ for some $a \in \Gamma$, $a \neq 1$.

The subgroup Γ acts in a natural way on the plane $\mathbb{R}^2$. We denote by $\Gamma\overline{v}$ the orbit of a vector $\overline{v} \neq 0$ under this action.

Lemma 6.5. *If $\overline{v}$ is a parabolic vector for Γ, then the orbit $\Gamma\overline{v}$ is discrete.*

Proof. So, $a\overline{v} = \overline{v}$ for some $a \in \Gamma$, $a \neq 1$. We assume that $\gamma_i\overline{v} \to \overline{w}$ as $i \to \infty$ for a sequence $\{\gamma_i\} \subset \Gamma$ and some $\overline{w} \in \mathbb{R}^2$. We put $a_i = \gamma_i a \gamma_i^{-1}$, $\overline{v}_i = \gamma_i\overline{v}$, $\overline{u}_i = \gamma_i\overline{u}$, where $\overline{u}$ is a vector orthogonal to $\overline{v}$. We find that $a_i\overline{v}_i = \overline{v}_i$, $a_i\overline{u}_i = \overline{u}_i + \alpha\overline{v}_i$, $\alpha \in \mathbb{R}$. Since the lengths of the vectors $\overline{v}_i$ are bounded above and all γ_i preserve area, the length of the projection of $\overline{u}_i$ on the direction orthogonal to $\overline{v}_i$ is bounded below by a positive constant. This implies that the sequence $\{a_i\}$ is bounded in $\mathrm{SL}(2,\mathbb{R})$. Since Γ is discrete, there are only finitely many distinct a_i. Without loss of generality

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we may assume that all a_i coincide. Since $a_i \neq 1$ and $a_i\overline{v}_i = \overline{v}_i$, we find that the vectors $\overline{v}_i$ are collinear. We put $b_i = \gamma_{i+1}\gamma_i^{-1}$; then $b_i\overline{v}_i = \overline{v}_{i+1} = \lambda_i\overline{v}_i$, with $\lambda_i \neq 0$. If $|\lambda_i| < 1$, the sequence of operators $b_i^n a_i b_i^{-n}$ tends to the identity operator as $n \to +\infty$; if $|\lambda_i| > 1$, we have to take $n \to -\infty$. Both cases contradict the discreteness of Γ, hence $|\lambda_i| = 1$. Then all terms of the sequence $\{\gamma_i\overline{v}\}$, from some term onwards, coincide (with $\overline{w}$), which proves that $\Gamma\overline{v}$ is discrete.

Lemma 6.6. *Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$. If $\overline{v}$ is a parabolic vector for Γ, then the orbit $\Gamma\overline{v}$ is discrete, otherwise 0 is a limit point of it. The number of discrete orbits, considered up to multiplication by a scalar, is finite.*

Proof. Let $\overline{v}$ be a parabolic vector for Γ, that is, $a\overline{v} = \overline{v}$ for some $a \in \Gamma$, $a \neq 1$. Since Γ is a lattice, a is conjugate to a power of one of a finite number of elements $a_1, \dots, a_k \in \Gamma \setminus \{1\}$ (see [16]). So, $\gamma a \gamma^{-1} = a_i^n$ for some $\gamma \in \Gamma$, $n \in \mathbb{Z}$ and some i, $1 \leqslant i \leqslant k$. Then $\gamma\overline{v} \in \Gamma\overline{v}$ is an eigenvector of a_i. So, the orbit of a parabolic vector coincides, up to multiplication by a scalar, with one of the orbits $\Gamma\overline{v}_1, \dots, \Gamma\overline{v}_k$, where $\overline{v}_1, \dots, \overline{v}_k$ are the eigenvectors of the operators $a_1, \dots, a_k$. Lemma 6.5 asserts that these orbits are discrete.

Now, let $\overline{v}$ be a non-parabolic vector for Γ. Without loss of generality we may assume that $\overline{v}$ is vertical (in fact, let $b \in \mathrm{SL}(2,\mathbb{R})$ be the operator mapping $\overline{v}$ to a vertical vector; then $b\overline{v}$ is a non-parabolic vector for the lattice $\Gamma_1 = b\Gamma b^{-1}$, and $\Gamma_1(b\overline{v}) = b(\Gamma\overline{v})$). In this case, by Lemma 3.7, $g^{t_i}\gamma_i \to h$ as $i \to \infty$, where $\{t_i\}$ is some sequence of numbers tending to $+\infty$, $g^t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, $\{\gamma_i\} \subset \Gamma$, $h \in \mathrm{SL}(2,\mathbb{R})$. Hence, $\gamma_i^{-1}g^{-t_i}\overline{v} \to h\overline{v}$ as $i \to \infty$. The vector $\overline{v}$ is vertical, so $g^{-t_i}\overline{v} = e^{t_i/2}\overline{v}$. Since $e^{t_i/2} \to \infty$ as $i \to \infty$, the sequence $\{\gamma_i^{-1}\overline{v}\}$ tends to the zero vector. Moreover, all its terms belong to the orbit $\Gamma\overline{v}$.

Lemma 6.7. *A discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ is a lattice if and only if we can choose finitely many parabolic vectors $\overline{v}_1, \dots, \overline{v}_k$ in it and for each $a \in \mathrm{SL}(2,\mathbb{R})$ we can find a vector $\overline{v} \in \Gamma\overline{v}_i$, $1 \leqslant i \leqslant k$, such that $|a\overline{v}| \leqslant C$, where C is a constant independent of a.*

Proof. Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$. We will use the notations introduced in the proof of Lemma 3.7. Let $\overline{v}_i$ $(1 \leqslant i \leqslant k)$ be the image of a horizontal vector under the action of the rotation r_i. Then $a_i\overline{v}_i = \pm\overline{v}_i$, $a_i^2\overline{v}_i = \overline{v}_i$, and $a_i^2 \in \Gamma$, $a_i^2 \neq 1$, that is, $\overline{v}_i$ is a parabolic vector for Γ.

We fix a point $z_0 \in \mathbb{H}^2$. For any $a \in \mathrm{SL}(2,\mathbb{R})$ we can find a $\gamma \in \Gamma$ such that the point $(\gamma a)z_0$ belongs to the fundamental polygon D. Then $(\gamma a)z_0$ belongs to the compact set K or to a wedge $r_i(\Pi(c_i;\alpha_i,\beta_i))$. In the latter case a power of a_i maps the point $(\gamma a)z_0$ into the wedge $r_i(\Pi(c_i;0,\beta_i - \alpha_i))$. Any point of this wedge can be mapped into the compact set $r_i(\widetilde{\Pi}(c_i;0,\beta_i - \alpha_i))$ by means of the operator $g_i^{-t} = r_i g^{-t} r_i^{-1}$, where $t \geqslant 0$ depends on the point. Thus, for any $a \in \mathrm{SL}(2,\mathbb{R})$ we can find a $\gamma \in \Gamma$, an index i, $1 \leqslant i \leqslant k$, and a number $t \geqslant 0$ such that $(g_i^{-t}\gamma a)z_0$ belongs to a compact set $\widetilde{K} \subset \mathbb{H}^2$ which does not depend on a. Consequently, $a = \gamma^{-1}g_i^t s$, where s belongs to the set $K_1 \subset \mathrm{SL}(2,\mathbb{R})$ of operators mapping z_0 into $\widetilde{K}$ (clearly, K_1 is compact). Replacing a by a^{-1} in the above reasoning, we find that a can be written in the form $a = s^{-1}g_i^{-t}\gamma$, with $\gamma \in \Gamma$, $s \in K_1$, $1 \leqslant i \leqslant k$, $t \geqslant 0$. We put $\overline{v} = \gamma^{-1}\overline{v}_i$. Then $\overline{v} \in \Gamma\overline{v}_i$ and $a\overline{v} = s^{-1}g_i^{-t}\overline{v}_i = s^{-1}(e^{-t/2}\overline{v}_i) = e^{-t/2}(s^{-1}\overline{v}_i)$. Since s lies in a compact subset of $\mathrm{SL}(2,\mathbb{R})$, the quantity $|s^{-1}\overline{v}_i|$ is bounded by a constant C which does not depend on s and i, that is, $|a\overline{v}| = e^{-t/2}|s^{-1}\overline{v}_i| \leqslant e^{-t/2}\cdot C \leqslant C$. This proves one assertion in the lemma.

We now assume that Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ for which we can choose finitely many parabolic vectors $\overline{v}_1,\dots,\overline{v}_k$ in accordance with the conditions of the lemma. Let $a_1,\dots,a_k$ be non-identity operators in Γ such that $a_i\overline{v}_i = \overline{v}_i$, $1 \leqslant i \leqslant k$. Let r_i ($1 \leqslant i \leqslant k$) be the rotation operators mapping the horizontal vector to $\overline{v}_i$; then $r_i^{-1}a_ir_i = \begin{pmatrix} 1 & \alpha_i \\ 0 & 1 \end{pmatrix}$, with $\alpha_i \neq 0$. By requirement, for each $a \in \mathrm{SL}(2,\mathbb{R})$ we can find a $\gamma \in \Gamma$ and an index i, $1 \leqslant i \leqslant k$, such that $|a\gamma\overline{v}_i| \leqslant C$, where C is a constant. We denote by bu_i a vector orthogonal to $\overline{v}_i$ and of the same length. Clearly, $a_i\overline{u}_i = \overline{u}_i \pm \alpha_i\overline{v}_i$, so for some $n \in \mathbb{Z}$ the length of the projection of the vector $a\gamma a_i^n\overline{u}_i$ on the direction of the vector $a\gamma a_i^n\overline{v}_i = a\gamma\overline{v}_i$ does not exceed $|\alpha_i|\cdot|a\gamma\overline{v}_i|$. We put $g_i^t = r_ig^tr_i^{-1}$; then $g_i^t\overline{v}_i = e^{t/2}\overline{v}_i$, $g_i^t\overline{u}_i = e^{-t/2}\overline{u}_i$, and for some $t \geqslant 0$ we have $|a\gamma a_i^ng_i^t\overline{v}_i| = C$. Moreover, the length of the projection of $a\gamma a_i^ng_i^t\overline{u}_i$ on the direction of the vector $a\gamma a_i^ng_i^t\overline{v}_i$ does not exceed $e^{-t/2}\cdot|\alpha_i|\cdot|a\gamma\overline{v}_i| \leqslant |\alpha_i|\cdot C$. On the other hand, the length of its projection on the orthogonal direction is $|\overline{v}_i|\cdot|\overline{u}_i|/C = |\overline{v}_i|^2/C$, since $a\gamma a_i^ng_i^t \in \mathrm{SL}(2,\mathbb{R})$. Thus, $|a\gamma a_i^ng_i^t\overline{u}_i| \leqslant \left(|\alpha_i|^2C^2 + |\overline{v}_i|^2/C^2\right)^{1/2}$, which does not exceed some constant. Consequently, $a\gamma a_i^ng_i^t$ belongs to a compact subset of $\mathrm{SL}(2,\mathbb{R})$.

What was said above implies that each $a \in \mathrm{SL}(2,\mathbb{R})$ can be written as $a = \gamma g_i^ts$, with $\gamma \in \Gamma$, $1 \leqslant i \leqslant k$, $t \geqslant 0$, and s belonging to the compact set $K_1 \subset \mathrm{SL}(2,\mathbb{R})$. We fix a point $z_0 \in \mathbb{H}^2$. For any $s \in K_1$ the point sz_0 belongs to the compact set $K \subset \mathbb{H}^2$, which does not depend on s. We now choose $c > 0$ so small that $K \subset \bigcap_{j=1}^k r_i(\Pi(c))$. Since $g_i^tz \in r_i(\Pi(c))$ for $t \geqslant 0$, if $z \in r_i(\Pi(c))$ we find that $\gamma^{-1}(az_0) = g_i^t(sz_0) \in r_i(\Pi(c))$. An arbitrary point $z \in \mathbb{H}^2$ can be written as az_0, $a \in \mathrm{SL}(2,\mathbb{R})$, so, under the action of an element of Γ it is mapped into one of the sets $r_i(\Pi(c))$. Applying now some power of a_i we can map this point into the wedge $r_i(\Pi(c;0,|\alpha_i|))$. The non-Euclidean area of each wedge is finite (see [16]). In view of the discreteness of Γ there is thus a fundamental domain of finite area for its action on $\mathbb{H}^2$, that is, Γ is a lattice.

Theorem 6.8. *A planar structure has property* B *if and only if its stabilizer is a lattice.*

Proof. The group of affine automorphisms of the planar structure ω acts in a natural manner on the set of saddle connections and on the set of ω-triangles.

Lemma 6.9. *If the stabilizer $\Gamma(\omega)$ of a planar structure ω is a lattice, then the number of orbits of the action on each of the sets listed above is finite.*

Proof. Let $\overline{v}$ be a vector in $\mathbb{R}^2$ that is a development of a saddle connection of ω. Then for any $a \in \Gamma(\omega)$ the vector $a\overline{v}$ is also a development of a saddle connection. Since there do not exist arbitrarily short saddle connections, 0 cannot be a limit

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Proof. If the pencils P_i
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Let Π' and Π'' be
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bijection $f\colon \Pi' \to \Pi''$ su
adjacent to $f(p)$ if p_1

point for the orbit $\Gamma(\omega)\overline{v}$, and by Lemma 6.6, $a\overline{v} = \overline{v}$ for some $a \in \Gamma(\omega)$, $a \neq 1$. In addition we can choose finitely many vectors $\overline{v}_1, \dots, \overline{v}_n$, independent of $\overline{v}$, such that the orbit $\Gamma(\omega)\overline{v}$ contains a vector parallel to one of these. Thus, using an affine automorphism, any saddle connection of ω can be mapped to a saddle connection parallel to one of the vectors $\overline{v}_1, \dots, \overline{v}_n$. There are only finitely many such saddle connections.

We now consider an arbitrary ω-triangle T. By an affine automorphism of the planar structure it can be transformed to an ω-triangle T_1 with one of its sides, denoted by L, parallel to some $\overline{v}_i$ ($1 \leqslant i \leqslant n$). Let $a_i \neq 1$ be an element of $\Gamma(\omega)$ for which $a_i\overline{v}_i = \overline{v}_i$. Applying to T_1 some power of the affine automorphism with linear part a_i, we obtain an ω-triangle T_2 with side L for which one of the angles at this side is larger than a certain constant $\alpha_i > 0$. This constant depends on a_i. Since the area of T_2 is bounded (by the area of M), this implies that the lengths of the sides of T_2 are bounded by a constant depending on L. Since L is one of finitely many saddle connections, an arbitrary ω-triangle can be transformed by an affine automorphism to an ω-triangle whose side lengths are bounded by a constant. There are only finitely many such ω-triangles.

Since the area of an ω-triangle does not change under an affine automorphism, the above lemma immediately implies that a planar structure with a lattice stabilizer has the property B.

We now turn to the second part of the assertion of the theorem. We assume that the planar structure ω has the property B. We choose an arbitrary direction $\overline{v}$ parallel to a saddle connection. By Proposition 6.1, the flow in the direction $\overline{v}$ splits into finitely many periodic pencils $P_1, \dots, P_k$. Let $w_1, \dots, w_k$ be the widths of these pencils.

Lemma 6.10. *The sequence $w_1, \dots, w_k$ coincides, up to multiplication of all terms by a scalar, with one of finitely many sequences, independent of $\overline{v}$.*

Proof. If the pencils P_i and P_j are adjacent (have a common saddle connection on their boundary), then w_i/w_j is the ratio of the areas of certain ω-triangles, so it can take only finitely many values. Further, the pencils $P_1, \dots, P_k$ can be ordered in such a way that each of them, from the second onwards, is adjacent to at least one of the previous pencils. Hence by induction on k we find that the sequence $w_1, \dots, w_k$ coincides, up to multiplication by a scalar, with one of finitely many sequences. To finish the proof we need to note that the number k of pencils is bounded by a constant not depending on $\overline{v}$.

We now consider a pair of directions $(\overline{v}_1, \overline{v}_2)$ parallel to saddle connections and giving the standard orientation in $\mathbb{R}^2$. The pencils of periodic trajectories parallel to $\overline{v}_1$ and $\overline{v}_2$ intersect, giving parallelograms. We will say that the parallelogram p_1 is *right-*, *left-*, *up-*, or *down-*adjacent to a parallelogram p if we can go from p to p_1 by moving along the vectors $\overline{v}_1$, $-\overline{v}_1$, $\overline{v}_2$, or $-\overline{v}_2$.

Let Π' and Π'' be the sets of parallelograms corresponding to the pairs of directions $(\overline{v}_1', \overline{v}_2')$ and $(\overline{v}_1'', \overline{v}_2'')$. We call such pairs *equivalent* if there is a bijection $f\colon \Pi' \to \Pi''$ such that the parallelogram $f(p_1)$ is right-, left-, up-, or down-adjacent to $f(p)$ if p_1 is adjacent to p with respect to the corresponding side.

The pairs $(\overline{v}_1', \overline{v}_2')$ and $(\overline{v}_1'', \overline{v}_2'')$ are called *strongly equivalent* if the bijection f can be chosen such that a parallelogram with sides w_1, w_2 becomes a parallelogram with sides αw_1, βw_2, where α and β are constants.

We assume that a pair of directions $(\overline{v}_1, \overline{v}_2)$ is such that we can find saddle connections L_1 and L_2 parallel to these directions that are two of the sides of an ω-triangle. The proof of Proposition 6.1 then implies that the area of each parallelogram corresponding to $(\overline{v}_1, \overline{v}_2)$ is bounded below by a constant $C > 0$ which depends on ω only. This implies that the number of parallelograms does not exceed S/C, where S is the area of M. Clearly, $(\overline{v}_1, \overline{v}_2)$ is now equivalent to one of finitely many pairs. Furthermore, Lemma 6.10 implies that this pair is also strongly equivalent to finitely many pairs.

The proof that $\Gamma(\omega)$ is a lattice can now be obtained from Lemma 6.7. In fact, by Lemmas 3.8, 3.9 and Proposition 6.2, the parabolic directions of $\Gamma(\omega)$ are precisely the directions parallel to saddle connections. Furthermore, for any parabolic direction $\overline{v}_1$ we can find a direction $\overline{v}_2$ such that the pair $(\overline{v}_1, \overline{v}_2)$ gives the standard orientation in $\mathbb{R}^2$ and there exist saddle connections L_1 and L_2 parallel to $\overline{v}_1$ and $\overline{v}_2$ that are sides of a single ω-triangle. What was said above implies that the pair $(\overline{v}_1, \overline{v}_2)$ is strongly equivalent to some pair $(\overline{v}_1^{(i)}, \overline{v}_2^{(i)})$ from a finite number of pairs $(\overline{v}_1^{(1)}, \overline{v}_2^{(1)}), \dots, (\overline{v}_1^{(m)}, \overline{v}_2^{(m)})$. The definition of strong equivalence now implies that there is an element $\gamma \in \Gamma(\omega)$ mapping $\overline{v}_1$ to $\overline{v}_1^{(i)}$ and $\overline{v}_2$ to $\overline{v}_2^{(i)}$. Thus, the orbit $\Gamma(\omega)\overline{v}_1$ of a parabolic vector coincides, up to multiplication by a scalar, with one of the orbits $\Gamma(\omega)\overline{v}_1^{(i)}$, $1 \leqslant i \leqslant m$. This implies that the sequence $\mathrm{SC}(\omega)$ consists of finitely many orbits. Using this and Proposition 3.2, on the basis of Lemma 6.7 we may conclude that $\Gamma(\omega)$ is a lattice.

Since property A is weaker than property B, Theorems 6.4 and 6.8 imply Veech's theorem.

Proposition 6.2 and Theorem 6.8 indicate that, probably, property A implies property B. It would be interesting to show this. If, however, this conjecture is not true, the construction of a corresponding counterexample (that is, of a planar structure having property A but not property B) would be no less interesting.

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