On Laplace transform of periodic functions

Recall that a function $f(t)$ is said to be *periodic of period* T if $f(t+T) = f(t)$ for all t. The goal of this handout is to prove the following (I even give two different proofs here).

Theorem 1. *If $f(t)$ is periodic with period T and piecewise continuous on the interval $[0, T]$, then the Laplace transform $F(s)$ of $f(t)$ satisfies:*

$$(1) \qquad F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) \, dt$$

Proof. Write the Laplace transform of $f(t)$ as two integrals:

$$(2) \qquad F(s) = \int_0^\infty f(t) e^{-st} \, ds = \int_0^T f(t) e^{-st} \, ds + \int_T^\infty f(t) e^{-st} \, ds.$$

Since $f(t - T) = f(t)$ for $t \geq T$, the second integral can be represented as follows:

$$\int_T^\infty f(t) e^{-st} \, dt = \int_T^\infty f(t - T) e^{-st} \, dt$$

.

Now substitute $\tau = t - T$. Then $t = \tau + T$, $d\tau = dt$ and τ runs from 0 to ∞. Therefore

$$\int_T^\infty f(t - T) e^{-st} \, dt = \int_0^\infty f(\tau) e^{-s(\tau + T)} \, d\tau = e^{-sT} \int_0^\infty f(\tau) e^{-s\tau} \, d\tau = e^{-sT} F(s)$$

Substituting this into (2) we get

$$F(s) = \int_0^T f(t) e^{-st} \, ds + e^{-sT} F(s) \Rightarrow (1 - e^{-sT}) F(s) = \int_0^T f(t) e^{-st} \, ds$$

which implies (1). $\qquad \square$

Remark 1. In some textbooks another proof of this theorem is given:

Let

$$g(t) = \begin{cases} f(t), & 0 \le\, t < T, \\ 0, & \text{otherwise.} \end{cases}$$

Then from the periodicity of F:

$$f(t) = g(t) + u_T(t) g(t - T) + u_{2T}(t) g(t - 2T) + u_{3T}(t) g(t - 3T) + \ldots$$

Then it is claimed (usually without any justification) that

$$(3) \qquad \mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} + \mathcal{L}\{u_T g(t - T)\} + \mathcal{L}\{u_{2T} g(t - 2T)\} + \mathcal{L}\{u_{3T} g(t - 3T)\} + \ldots$$

The claim is true but formally you have to justify it (here you have an infinite sum, so it does not formally follow from the linearity of Laplace transform).

Further by the translation in t property

$$\mathcal{L}\{u_{kT}(t) g(t - kT)\} = e^{-kTs} G(s),$$

where $G(s) = \mathcal{L}\{g(t)\}(s) = \int_0^T e^{-st} f(t) \, dt$. From this and 3 it follows that

$$(4) \qquad \mathcal{L}\{f(t)\} = (1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots) \int_0^T e^{-st} f(t) \, dt$$

Now recall that $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ for $|x| < 1$ (the formula for the infinite sum of geometric progression). Therefore, $(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots) = \frac{1}{1 - e^{-sT}}$. Substituting this to (4) we get the required formula (1).