Homework Assignment # 10 Solution, MATH 308, FALL 2013

1. (a) Determine $\omega_0 > 0$, $R > 0$ and $\delta \in [0, 2\pi)$ so as to write the expression $-3\cos 7t + 4\sin 7t$ in the form $R\cos(\omega_0 t - \delta)$; (you can use a calculator to determine an approximate value of δ);

We know that

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$$

where $R = \sqrt{A^2 + B^2}$

$$\tan\delta = \frac{B}{A}$$

In our case $\underbrace{A = -3, \quad B = 4,}$ $\boxed{\omega_0 = 7}$ $\Rightarrow \frac{\pi}{2} < \delta < \pi$

$$R = \sqrt{(-3)^2 + 4^2} = \boxed{5}$$

$$\Downarrow$$

$$\tan\delta = \frac{4}{-3} \Rightarrow \delta = \pi - \arctan\left(\frac{4}{3}\right) = 180^\circ - 53^\circ = 127^\circ$$

Finally,

-3cos(7t)+4sin(7t)=5cos(7t-127π/180)

(b) A mass weighing 16 lb is attached to a 5 ft-long spring. At equilibrium the spring measures 8.2 ft. Assume that there is no damping. If after this the mass is pushed 2 ft down and then set in motion with downward velocity of 4 ft/s, determine the position u of the mass at any time t.

Given:
$mg=16$
$l=5$
$l+L=8.2$
$\gamma=0$
$u(0)=2$
$u'(0)=4$

Find $u(t)$

$L=8.2-l=8.2-5=3.2$ ft
$m=16/g=16/32=\frac{1}{2}$ slug
$mg=kL$, i.e. $16=3.2k$, or **$k=5$**

We know that the position of undamped unforced vibration satisfies the following ODE $mu''+ku=0$. In our case this means

$\frac{1}{2}u''+5u=0$, or $u''+10u=0$.

Characteristic equation: $r^2+10=0$, i.e. $r_{1,2}=\pm i\sqrt{(10)}$
Fundamental set: $\{\cos(\sqrt{10}\,t),\ \sin(\sqrt{10}\,t)\}$

Genearl solution

$$u(t)=C_1\cos(\sqrt{10}\,t)+C_2\sin(\sqrt{10}\,t)$$

Solution subject to the initial conditions:

$$u(0)=\boxed{C_1=2}$$

$$u'(t)=-\sqrt{10}\,C_1\sin(\sqrt{10}\,t)+C_2\sqrt{10}\cos(\sqrt{10}\,t)$$

$$u'(0)=C_2\sqrt{10}=4\Rightarrow C_2=\frac{4}{\sqrt{10}}$$

Finally, $$u(t)=2\cos\sqrt{10}\,t+\frac{4}{\sqrt{10}}\sin\sqrt{10}\,t$$

(c) Find the natural frequency, the period, the amplitude, and the phase of the motion of the spring-mass system of item (b) (you can use calculator to determine the phase).

$$u(t) = 2\cos\sqrt{10}\,t + \frac{4}{\sqrt{10}}\sin\sqrt{10}\,t$$

We have: ***$A=2$, $B=4*10^{-1/2}$***

Thus the amplitude ***$R=(A^2+B^2)^{1/2}=(4+16/10)^{1/2}=(28/5)^{1/2}$***

natural frequency ***$\omega_0=10^{1/2}$***

Period ***$T=2\pi/10^{1/2}$***

To find phase note that both A and B are positive, i.e. the phase angle is in the first qudrant and it can be found as ***$\delta=\arctan(B/A)=\arctan(4*10^{-1/2}/2)=32^o$***, or ***0.56 radian***

(d) Assume that in the case of the spring-mass system of item (b) there is also a damping and we can change the damping constant. What is the critical damping constant?

We know that $\gamma_{crit} = 2\sqrt{km} = 2(5/2)^{1/2}$

2 A mass weighing 32 lb stretches a spring $\frac{8}{3}$ ft. The mass is initially released from rest from a point 2 ft below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to the instantaneous velocity. If the mass is driven by an external force $F(t) = 20\cos(3t)$, then

(a) Find the equation of motion.

Given:
$mg=32$
$L=8/3$
$u'(0)=0$
$u(0)=2$
$\gamma=1$
$F(t) = 20\cos(3t)$

We know that the position satisfies the following ODE

$$mu'' + \gamma u' + ku = F(t).$$

In our case: $mg=kL$ implies $k=32/(8/3)=12$
and also $m=32/g=32/32=1$

and we have ODE:
$u''+u'+12u=20\cos(3t)$ subject to the initial conditions
$u(0)=2,\ u'(0)=0$

To determine the equation of motion we seek thee general solution in the form $u(t)=u_h(t)+u_p(t)$.

First solve the corresponding homogeneous ODE: $\mathbf{u''+u'+12u=0.}$

The caracteristic equation $r^2+r+12=0$
has the following roots $r_{1,2}=(-1\pm\sqrt{(-47)})/2=(-1\pm\sqrt{(47)}i)/2$

The corresponding general solution is

$$u_h=e^{-t/2}(C_1\cos(\tfrac{1}{2}\sqrt{(47)}t)+C_2\sin(\tfrac{1}{2}\sqrt{(47)}t))$$

To find a particular solution we apply the Method of Undetermined Coefficients.
From the form of the external force $F(t)=20\cos(3t)$ it follows that

$$\alpha+i\beta=3i\neq r_{1,2}.$$

Thus the multiplicity $s=0$ and then we find the a particular solution in the form

$$u_p(t)=t^s(A\cos(3t)+B\sin(3t)),$$

or

$$u_p(t)=u(t)=A\cos(3t)+B\sin(3t)$$

Find A and B:
$u=A\cos(3t)+B\sin(3t)$
$u'=-3A\sin(3t)+3B\cos(3t)$
$u''=-9A\cos(3t)-9B\sin(3t)$

In our case we have
$12u=12A\cos(3t)+12B\sin(3t)$
$+$
$u'=-3A\sin(3t)\ +3B\cos(3t)$
$+$
$u''=-9A\cos(3t)\ -9B\sin(3t)$
$=$
$20\cos(3t)=(12A+3B-9A)\cos(3t)+(12B-3A-9B)\sin(3t),$
or
$20\cos(3t)=(3A+3B)\cos(3t)+(3B-3A)\sin(3t)$
which implies
$3A+3B=20$
$3A-3B=0.$ *The solution is* $A=B=10/3$. *Thus, a particular solution is*

$$u_p(t)=10(\cos(3t)+\sin(3t))/3$$

General solution will be $u(t)=u_h(t)+u_p(t)$, *or*

$$u(t)=e^{-t/2}(C_1\cos(\tfrac{1}{2}\sqrt{(47)}t)+C_2\sin(\tfrac{1}{2}\sqrt{(47)}t))+10(\cos(3t)+\sin(3t))/3.$$

It remains to solve the corresponding IVP, i.e. to determine C_1 and C_2. Given
$u(0)=2$, or $C_1+10/3=2$, $\quad C_1=2-10/3=-4/3$
$u'(0)=0$

$$u'(t)=-\tfrac{1}{2}e^{-t/2}(C_1\cos(\tfrac{1}{2}\sqrt{(47)}t)+C_2\sin(\tfrac{1}{2}\sqrt{(47)}t))+e^{-t/2}(-\tfrac{1}{2}\sqrt{(47)}C_1\sin(\tfrac{1}{2}\sqrt{(47)}t)+\tfrac{1}{2}\sqrt{(47)}C_2\cos(\tfrac{1}{2}\sqrt{(47)}t))+10(-\sin(3t)+\cos(3t)).$$

Hence, $u'(0)=-\tfrac{1}{2}C_1+\tfrac{1}{2}\sqrt{(47)}C_2+10=0$. *Substitute* **$C_1=-4/3$ and conclude that $C_2=-64/(3\sqrt{(47)})$.**

Finally, the equation of motion is

$$u(t)=e^{-t/2}((-4/3)\cos(\tfrac{1}{2}\sqrt{(47)}t)-\mathbf{64/(3\sqrt{(47)})}\sin(\tfrac{1}{2}\sqrt{(47)}t))+10(\cos(3t)+\sin(3t))/3.$$

(b) Determine the steady state solution of this system.

Steady state solution in this case is $u_p(t)$, *or*
$u(t)=10(\cos(3t)+\sin(3t))/3$

Remark: in alternative form the steady state solution will be
$u(t)=(10(\sqrt{2})/3)\cos(3t-\pi/4)$