Homework # 11 Solutions MATH 308-505 FALL 2013

1. Using an appropriate algebra and Laplace Transform properties (see also the Table) find the Laplace Transform of the given functions:

(a) $f(t) = (2013 + e^{-t} - 3e^{2t}) \sin 4t$

$$f(t) = 2013 \sin 4t + e^{-t} \sin 4t - 3e^{2t} \sin 4t$$

$$F(s) = \mathcal{L}\{f(t)\} = 2013\mathcal{L}\{\sin 4t\} + \mathcal{L}\{e^{-t}\sin 4t\} - 3\mathcal{L}\{e^{2t}\sin 4t\}$$

Using the Table

$$F(s) = \frac{2013 \cdot 4}{s^2+16} + \frac{4}{(s+1)^2+16} - \frac{3 \cdot 4}{(s-2)^2+16}$$

$$F(s) = \frac{8052}{s^2+16} + \frac{4}{(s+1)^2+16} - \frac{12}{(s-2)^2+16}$$

1 (b) $g(t) = e^{13t}(t+1)^2 = e^{13t}(t^2+2t+1) = e^{13t}t^2 + 2e^{13t}t + e^{13t}$

$$G(s) = \mathcal{L}\{g(t)\} = \frac{2!}{(s-13)^{2+1}} + 2\frac{1!}{(s-13)^{1+1}} + \frac{1}{s-13}$$

$$G(s) = \frac{2}{(s-13)^3} + \frac{2}{(s-13)^2} + \frac{1}{s-13}$$

1 (c) $y(t) = e^t(4 + 5t^3 + 12\cos\frac{t}{4}) = 4e^t + 5t^3e^t - 12e^t\cos\frac{t}{4}$

$Y(s) = \mathcal{L}\{y(t)\} = 4\,\mathcal{L}\{e^t\} + 5\,\mathcal{L}\{t^3e^t\} - 12\,\mathcal{L}\{e^t\cos\frac{t}{4}\}$

$Y(s) = \frac{4}{s-1} + 5\frac{3!}{(s-1)^{3+1}} - 12\frac{s-1}{(s-1)^2 + (\frac{1}{4})^2}$

$Y(s) = \frac{4}{s-1} + \frac{30}{(s-1)^4} - \frac{12(s-1)}{(s-1)^2 + \frac{1}{16}}$

Problem 2 a

$$F(s) = \frac{2s+1}{s^2-7s+12}$$

$s^2-7s+12=0$

$D = 7^2-4\cdot 12 = 49-48 = 1$

$s_1 = \frac{7+1}{2} = 4 \qquad \Rightarrow (s^2-7s+12) = (s-3)(s-4)$

$s_2 = \frac{7-1}{2} = 3$

The partial fraction decomposition is of the form

$$\frac{2s+1}{(s-3)(s-4)} = \frac{A}{s-3} + \frac{B}{s-4} \Rightarrow$$

$$2s+1 = A(s-4) + B(s-3)$$

To find A put $s=3$: $2\cdot 3+1 = A(3-4) \Rightarrow -A = 7 \Rightarrow A = -7$

To find B put $s=4$: $2\cdot 4+1 = B(4-3) \Rightarrow B = 9 \Rightarrow$

$$\frac{2s+1}{(s-3)(s-4)} = -\frac{7}{s-3} + \frac{9}{s-4} \Rightarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = -7\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 9\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = \boxed{-7e^{3t}+9e^{4t}}$$

Problem 2 B

$$H(s) = \frac{3s-9}{s^2+4s+29}$$

$s^2+4s+29 = s^2+4s+4+25 = (s+2)^2+5^2 \qquad (\alpha=-2, \beta=5)$

We look for the decomposition in the form $\frac{3s-9}{s^2+4s+29} = \frac{A(s+2)+5B}{(s+2)^2+25} \Rightarrow$

$3s-9 = A(s+2)+5B$. To find B put $s=-2$: $-6-9 = 5B \Rightarrow -15 = 5B \Rightarrow \boxed{B=-3}$

To find A compare coefficient of s: $3 = A \Rightarrow \boxed{A=3} \Rightarrow$

$$\frac{3s-9}{s^2+4s+29} = 3\frac{s+2}{(s+2)^2+5^2} - 3\frac{5}{(s+2)^2+5^2} \Rightarrow \boxed{\mathcal{L}^{-1}\{H(s)\} = 3e^{-2t}\cos 5t - 3e^{-2t}\sin 5t}$$

2 (c) $Y(s) = \frac{2s-1}{s^2(s+1)^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)^3}$

$2s-1 = As(s+1)^3 + B(s+1)^3 + Cs^2(s+1)^2 + Ds^2(s+1) + Es^2$

$2s-1 = As(s^3+3s^2+3s+1) + B(s^3+3s^2+3s+1)$
$+ Cs^2(s^2+2s+1) + Ds^2(s+1) + Es^2$

$s=0 \Rightarrow \boxed{-1 = B}$

$s=-1 \Rightarrow -3 = E \Rightarrow \boxed{E=-3}$

$s^4 : \; 0 = A + C$

$s^3 : \; 0 = 3A + B + 2C + D$

$s : \; 2 = A + 3B \Rightarrow A = 2 - 3B = 2 - 3\cdot(-1) \Rightarrow \boxed{A=5}$

$\rightarrow C = -A \Rightarrow \boxed{C=-5}$

$\rightarrow D = -(3A+B+2C) = -(15-1-10) = -4$

$\boxed{D=-4}$

$Y(s) = \frac{5}{s} + \frac{-1}{s^2} + \frac{-5}{s+1} + \frac{-4}{(s+1)^2} + \frac{-3}{(s+1)^3}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 5\,\mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{1}{s^2}\}$

$-5\mathcal{L}^{-1}\{\frac{1}{s+1}\} - 4\,\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} - 3\mathcal{L}^{-1}\{\frac{1}{(s+1)^3}\}$

Using the Table we conclude

$y(t) = 5 - t - 5e^{-t} - 4te^{-t} - \frac{3}{2}t^2e^{-t}$

3 Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem (you do not need to find the solution $y(t)$ itself here):

(a) $4y'' - 17y' + 13y = e^{-t}\cos 3t, \quad y(0) = 2,\ y'(0) = -1;$

$$\mathcal{L}\{4y'' - 17y' + 13y\} = \mathcal{L}\{e^{-t}\cos 3t\}$$

$$4(s^2Y(s) - sy(0) - y'(0)) - 17(sY(s) - y(0)) + 13Y(s) = \frac{s+1}{(s+1)^2 + 3^2}$$

$$4s^2Y(s) - 2s - (-1) - 17sY(s) + 17\cdot 2 + 13Y(s) = \frac{s+1}{(s+1)^2 + 9}$$

$$Y(s)[4s^2 - 17s + 13] \quad \frac{s+1}{(s+1)^2+9} + 2s - 35$$

$$Y(s) = \frac{s+1 + (2s-35)((s+1)^2 + 9)}{4s^2 - 17s + 13}$$

3 (b) $2y'' + 3y' - 5y = t^4 e^{4t}, \quad y(0) = 1,\ y'(0) = 0$

$$\mathcal{L}\{2y'' + 3y' - 5y\} = \mathcal{L}\{t^4 e^{4t}\}$$

$$2\left(s^2 Y(s) - s\,y(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) - 5Y(s)$$

$$= \frac{4!}{(s-4)^{4+1}}$$

$$2s^2 Y(s) - 2s + 3sY(s) - 3 - 5Y(s) = \frac{24}{(s-4)^5}$$

$$Y(s)\left[2s^2 + 3s - 5\right] = \frac{24}{(s-4)^5} + 2s + 3$$

$$Y(s) = \frac{24 + (2s+3)(s-4)^5}{2s^2 + 3s - 5}$$

Problem 4 -8

$4y'' - 17y' + 13y = e^{-t}\cos 3t \quad , y(0) = 0, \; y'(0) = 0$

By analogy with 3a (and taking into account that the initial conditions are different: $y(0) = y'(0) = 0$) we get

$$\underbrace{(4s^2 - 17s + 13)}_{(4s-13)(s-1)} Y(s) = \frac{s+1}{(s+1)^2 + 9} \Rightarrow$$

$$Y(s) = \frac{s+1}{((s+1)^2+9)(4s-13)(s-1)} = A\frac{s+1}{(s+1)^2+9} + B\frac{3}{(s+1)^2+9} + \frac{C}{4s-13} + \frac{D}{s-1}$$

$$\Rightarrow \quad s+1 = A(s+1)(4s-13)(s-1) + 3B(4s-13)(s-1) +$$

$$C((s+1)^2+9)(s-1) + D((s+1)^2+9)(4s-13)$$

To find D put $s = 1$: $1+1 = D((1+1)^2+9)(4-13) \Rightarrow$

$2 = 13 \cdot (-9) D \Rightarrow \boxed{D = -\frac{2}{117}}$

To find C put $s = \frac{13}{4} \Rightarrow \frac{13}{4} + 1 = C\left(\left(\frac{13}{4}+1\right)^2 + 9\right)\left(\frac{13}{4} - 1\right)$

$\frac{17}{\cancel{4}} = \left(\left(\frac{289}{16} + 9\right)\frac{9}{\cancel{4}}\right)C \Rightarrow 272 = 3897\,C \Rightarrow \boxed{C = \frac{272}{3897}}$

To find A compare coefficients of s^3: $0 = 4A + C + 4D \Rightarrow$

$A = -\frac{C}{4} - D = -\frac{68}{3897} + \frac{2}{117} = \boxed{-\frac{2}{5629}}$

To find B put $s = -1$: $0 = 3B \cdot (-17) \cdot (-2) + C \times 9 \times (-2) +$

$+ D \times 9(-17) = 102B - 18C - 153D$

$$\Rightarrow B = \frac{18}{102}C + \frac{153}{102}D = \frac{3}{17}C + \frac{3}{2}D = \frac{3}{\cancel{17}} \cdot \frac{\cancel{272}^{16}}{3897} + \frac{3}{2} \cdot \left(-\frac{2}{117}\right) =$$

$$= \frac{48}{3897} - \underbrace{\frac{3}{117}}_{\frac{1}{39}} = -\frac{75}{5629}$$

$$\Downarrow$$

$$Y(s) = -\frac{2}{5629}\frac{s+1}{(s+1)^2+9} - \frac{75}{5629}\frac{3}{(s+1)^2+9} + \underbrace{\frac{272}{3897}\frac{1}{4s-13}}_{\frac{68}{3897}\frac{1}{s-13/4}} - \frac{2}{117}\frac{1}{s-1}$$

$$\Downarrow$$

$$\boxed{y(t) = -\frac{2}{5629}e^{-t}\cos 3t - \frac{75}{5629}e^{-t}\sin 3t + \frac{68}{3897}e^{\frac{13}{4}t} - \frac{2}{117}e^{t}}$$