Homework Assignment #12 — Fall 2013 - MATH308

due Wednesday Oct 30 at the beginning of class

Topics covered: *step function and Laplace transform of discontinuous functions (corresponds to sections 6.3, 6.4 in the textbook)*

1. Find the Laplace transform of the function

$$f(t) = \begin{cases} 7 & t < 4, \\ -2t+7 & 4 \le t < 8, \\ t^2+2t & 8 \le t. \end{cases}$$

Solution : First write the given function with jump discontinuities in compact form:

$$f(t)=7+(-2t+7-7)u_4(t)+(t^2+2t-(-2t+7))u_8(t)=7-2tu_4(t)+(t^2+4t-7)u_8(t)$$

Now use linearity of Laplace Transform to get

$$\mathbf{F(s)=} \ 7L\{1\}-2L\{tu_4(t)\}+L\{(t^2+4t-7)u_8(t)\}=7s^{-1}-2L\{tu_4(t)\}+L\{(t^2+4t-7)u_8(t)\}.$$

Apply translation in t property $L\{g(t-c)u_c(t)\}=e^{-cs}G(s)$ to the remaining Laplace transforms we get :

$L\{tu_4(t)\}=L\{g(t-4)u_4(t)\}=e^{-4s}G(s)$, where $g(t-4)=t$, which implies $g(t)=t+4$, i.e. $G(s)=s^{-2}+4s^{-1}$. As a result,

$$L\{tu_4(t)\}=e^{-4s}(s^{-2}+4s^{-1}).$$

Next,

$L\{(t^2+4t-7)u_8(t)\}=L\{g(t-8)u_8(t)\}$, where $g(t-8)=t^2+4t-7$, which implies

$$g(t)=(t+8)^2+4(t+8)-7=t^2+16t+64+4t+32-7=t^2+20t+89,$$

i.e.

$$G(s)=2s^{-3}+20s^{-2}+89s^{-1}.$$

As a result,

$$L\{(t^2+4t-7)u_8(t)\}=e^{-8s}(2s^{-3}+20s^{-2}+89s^{-1}).$$

Finally,

$$F(s)=7s^{-1}-2e^{-4s}(s^{-2}+4s^{-1})+e^{-8s}(2s^{-3}+20s^{-2}+89s^{-1}).$$

2. Find the inverse Laplace transform of the function $\frac{e^{-2s}(2s+1)}{s^3-6s^2+13s} = e^{-2s}F(s)$

where

$$F(s) = \frac{2s+1}{s^3-6s^2+13s} = \frac{2s+1}{s(s^2-6s+13)}$$

$$F(s) = \frac{2s+1}{s((s-3)^2+4)} = \frac{A}{s} + \frac{B(s-3)+2C}{(s-3)^2+4}$$

$$f(t) = A\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + B\,\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+4}\right\} + C\,\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^2+4}\right\}$$

$$f(t) = A + B\,e^{3t}\cos(2t) + C\sin(2t)$$

Determine A, B, C

$$2s+1 = A((s-3)^2+4) + (B(s-3)+2C)\,s$$

$s=3$: $\quad 7 = 4A + 6C \Rightarrow C = \frac{7-4A}{6} = \frac{7-1}{6} \Rightarrow \boxed{C=1}$

$s=0$: $\quad 1 = 13A \Rightarrow \boxed{A = \frac{1}{13}}$

s^2: $\quad 0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B = -\frac{1}{13}}$

$$f(t) = \frac{1}{13} - \frac{1}{13}e^{3t}\cos(2t) + \sin(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}(2s+1)}{s^3-6s^2+13s}\right\} = \mathcal{L}^{-1}\{e^{-2s}F(s)\} = u_2(t)\,f(t-2)$$

$$= u_2(t)\left(\frac{1}{13} - \frac{1}{13}e^{3(t-2)}\cos 2(t-2) + \sin 2(t-2)\right)$$

3. Find the solution of the initial value problem $y'' + y = g(t)$; $y(0) = 0$, $y'(0) = 1$, where

$$g(t) = \begin{cases} \cos(4t), & 0 \le t < \pi \\ 0, & t \ge \pi. \end{cases}$$

Step 1

$$\mathcal{L}\{y''+y\} = s^2Y(s) - sy(0) - y'(0) + Y(s)$$
$$= s^2Y(s) - 1 + Y(s)$$
$$= (s^2+1)Y(s) - 1$$

Step 2 Write $g(t)$ in compact form

$$g(t) = \cos(4t) + (0 - \cos(4t))u_\pi(t)$$
$$= \cos(4t) - \cos(4t)u_\pi(t)$$

STEP 3

$$G(s) = \mathcal{L}\{\cos(4t)\} - \mathcal{L}\{\cos(4t)\, u_\pi(t)\}$$
$$= \frac{s}{s^2+16} - \mathcal{L}\{h(t-\pi)\, u_\pi(t)\}$$

where $h(t-\pi) = \cos 4t$

$h(t) = \cos 4(t+\pi) = \cos(4t+4\pi) = \cos 4t$

$$G(s) = \frac{s}{s^2+16} - e^{-\pi s}H(s) = \frac{s}{s^2+16} - e^{-\pi s}\frac{s}{s^2+16}$$

STEP 4 Combine Step 1 and Step 3:

$$(s^2+1)Y(s) - 1 = \frac{s}{s^2+16} - e^{-\pi s}\frac{s}{s^2+16}$$

$$Y(s) = \frac{s}{(s^2+16)(s^2+1)} - e^{-\pi s}\frac{s}{(s^2+16)(s^2+1)} + \frac{1}{s^2+1}$$

$$Y(s) = F(s) - e^{-\pi s}F(s) + \frac{1}{s^2+1}$$

where $F(s) = \dfrac{s}{(s^2+16)(s^2+1)}$

$$y(t) = f(t) - f(t-\pi)u_\pi(t) + \sin t$$

STEP 5 Find $f(t)$:

$$F(s) = \frac{s}{(s^2+16)(s^2+1)} = \frac{As+4B}{s^2+16} + \frac{Cs+D}{s^2+1}$$

$$s = (As+4B)(s^2+1) + (Cs+D)(s^2+16)$$

s^3: $0 = A + C \Rightarrow C = -A$

s^2: $0 = 4B + D$

s: $1 = A + 16C \rightarrow 1 = A - 16A \Rightarrow \boxed{A = -\frac{1}{15}}$

$C = \frac{1}{15}$

s^0: $0 = 4B + 16D$

$B = D = 0$

$$F(s) = -\frac{1}{15}\frac{s}{s^2+16} + \frac{1}{15}\frac{s}{s^2+1}$$

$$f(t) = -\frac{1}{15}\cos(4t) + \frac{1}{15}\cos t$$

Finally,

$$y(t) = f(t) - f(t-\pi)u_\pi(t) + \sin t$$

$$y(t) = -\frac{1}{15}\cos(4t) + \frac{1}{15}\cos t - \left[-\frac{1}{15}\cos(4(t-\pi)) + \frac{1}{15}\cos(t-\pi)\right]$$
$$\times u_\pi(t) + \sin t$$

OR

$$y(t) = -\frac{1}{15}\cos(4t) + \frac{1}{15}\cos t + \frac{1}{15}[\cos 4t - \cos t]u_\pi(t) + \sin t$$