due Monday Nov 4 at the beginning of class

Topics covered: *initial value problems with impulse forcing functions; convolution integrals (corresponds to sections 6.5, 6.6 in the textbook)*

1. Given IVP

$$y'' + 25y = \delta(t - \frac{3\pi}{2}) + 25\delta(t - \frac{\pi}{2}), \quad y(0) = 1, y'(0) = 0.$$

   (a) Solve the given IVP. Simplify your answers using trigonometric formulas (as it was demonstrated in class).

Apply Laplace Transform to LHS

$$\mathcal{L}\{y''+25y\} = s^2 Y(s) - sy(0) - y'(0) + 25Y(s)$$
$$= s^2 Y(s) - s\cdot 1 - 0 + 25Y(s)$$
$$= (s^2+25)Y(s) - s$$

Apply Laplace Transform to RHS

$$\mathcal{L}\{\delta(t-\tfrac{3\pi}{2}) + 25\delta(t-\tfrac{\pi}{2})\} = e^{-\frac{3\pi}{2}s} + 25e^{-\frac{\pi}{2}s}$$

As a result, we get

$$(s^2+25)Y(s) - s = e^{-\frac{3\pi}{2}s} + 25e^{-\frac{\pi}{2}s}$$
$$(s^2+25)Y(s) = e^{-\frac{3\pi}{2}s} + 25e^{-\frac{\pi}{2}s} + s$$
$$Y(s) = \frac{e^{-\frac{3\pi}{2}s}}{s^2+25} + \frac{25e^{-\frac{\pi}{2}s}}{s^2+25} + \frac{s}{s^2+25}$$

Denote $F(s) = \frac{1}{s^2+25}$ then

$$Y(s) = e^{-\frac{3\pi}{2}s}F(s) + 25e^{-\pi/2\, s}F(s) + \frac{s}{s^2+25}$$

$$y(t) = u_{\frac{3\pi}{2}}(t)\, f(t - \tfrac{3\pi}{2}) + 25u_{\frac{\pi}{2}}(t) f(t-\tfrac{\pi}{2}) + \cos(5t)$$

Find $f(t)$:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} = \frac{1}{5}\sin(5t)$$

Hence,

$$y(t) = u_{\frac{3\pi}{2}}(t)\cdot\frac{1}{5}\sin 5(t-\tfrac{3\pi}{2}) + 25u_{\frac{\pi}{2}}(t)\frac{1}{5}\sin 5(t-\tfrac{\pi}{2}) + \cos(5t)$$

or

$$y(t) = \frac{1}{5}u_{\frac{3\pi}{2}}(t)\sin(5t - \tfrac{15\pi}{2}) + 5u_{\frac{\pi}{2}}(t)\sin(5t-\tfrac{5\pi}{2}) + \cos(5t)$$

Note that $\sin(5t - \frac{15\pi}{2}) = \sin(5t - \frac{16\pi-\pi}{2})$

$$= \sin(5t - 8\pi + \tfrac{\pi}{2}) = \sin(5t+\tfrac{\pi}{2}) = \cos(5t)$$

and

$$\sin(5t - \tfrac{5\pi}{2}) = \sin(5t - \tfrac{4\pi+\pi}{2})$$
$$= \sin(5t - 2\pi - \tfrac{\pi}{2}) = \sin(5t - \tfrac{\pi}{2}) = -\cos(5t)$$

Finally,

$$y(t) = \frac{1}{5}u_{\frac{3\pi}{2}}(t)\cos 5t - 5u_{\frac{\pi}{2}}(t)\cos(5t) + \cos(5t)$$

or, equivalently,

$$y(t) = \left(\frac{1}{5}u_{\frac{3\pi}{2}}(t) - 5u_{\frac{\pi}{2}}(t) + 1\right)\cos(5t)$$

(b) Graph the obtained solution.

In part (a) we obtained

$$y(t) = \left(\frac{1}{5} u_{\frac{3\pi}{2}}(t) - 5 u_{\frac{\pi}{2}}(t) + 1\right)\cos(5t)$$

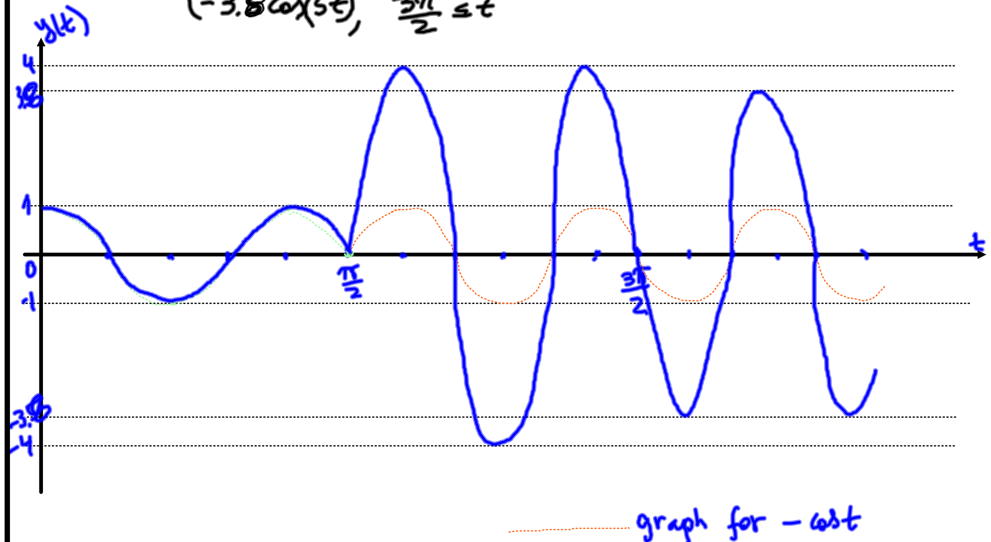
Represent $y(t)$ in alternative form, using that

$$u_{\frac{3\pi}{2}}(t) = \begin{cases} 0, & 0 \le t < \frac{3\pi}{2} \\ 1, & \frac{3\pi}{2} \le t \end{cases} \quad \text{and} \quad u_{\frac{\pi}{2}}(t) = \begin{cases} 0, & 0 \le t < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} \le t \end{cases},$$

$$y(t) = \begin{cases} \left(\frac{1}{5}\cdot 0 - 5\cdot 0 + 1\right)\cos(5t), & 0 \le t < \frac{\pi}{2} \\ \left(\frac{1}{5}\cdot 0 - 5\cdot 1 + 1\right)\cos(5t), & \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ \left(\frac{1}{5}\cdot 1 - 5\cdot 1 + 1\right)\cos(5t), & \frac{3\pi}{2} \le t \end{cases}$$

OR

$$y(t) = \begin{cases} \cos(5t), & 0 \le t < \frac{\pi}{2} \\ -4\cos(5t), & \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ -3.8\cos(5t), & \frac{3\pi}{2} \le t \end{cases}$$



2. Use the convolution theorem to find the inverse Laplace transform of the given function:

$$\frac{s}{(s^2+9)(s^2+25)}$$

Denote $F(s)=\frac{s}{s^2+9} \Rightarrow f(t)=\cos(3t)$

$G(s)=\frac{1}{s^2+25} \Rightarrow g(t)=\frac{1}{5}\sin(5t)$

Then

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)(s^2+25)}\right\}=\mathcal{L}^{-1}\{F(s)G(s)\}=f*g$$

$$=\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t \cos 3(t-\tau)\cdot\frac{1}{5}\sin 5\tau\,d\tau$$

$$=\frac{1}{5}\int_0^t \cos(3t-3\tau)\sin(5\tau)\,d\tau$$

Use trigonometric identity:

$$\cos A\,\sin B=\frac{1}{2}\left(\sin(B-A)+\sin(B+A)\right)$$

In our case

$A=3t-3\tau$, $B=5\tau$. Hence,

$B-A=5\tau-3t+3\tau=8\tau-3t$

$B+A=5\tau+3t-3\tau=2\tau+3t$

So,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)(s^2+25)}\right\}=\frac{1}{5}\int_0^t\frac{1}{2}\left(\sin(8\tau-3t)+\sin(2\tau+3t)\right)d\tau$$

$$=\frac{1}{10}\int_0^t\left(\sin(8\tau-3t)+\sin(2\tau+3t)\right)d\tau$$

$$=-\frac{1}{10}\left(\frac{1}{8}\cos(8\tau-3t)+\frac{1}{2}\cos(2\tau+3t)\right)\Big|_{\tau=0}^{t}$$

$$=-\frac{1}{10}\left(\frac{1}{8}\left(\cos(5t)-\cos(-3t)\right)+\frac{1}{2}\left(\cos(5t)-\cos(3t)\right)\right)$$

$$=-\frac{1}{10}\left[\frac{1}{8}\cos(5t)-\frac{1}{8}\cos(3t)+\frac{1}{2}\cos(5t)-\frac{1}{2}\cos(3t)\right]$$

$$=-\frac{1}{10}\left[\frac{5}{8}\cos(5t)-\frac{5}{8}\cos(3t)\right]$$

$$=\frac{1}{16}\left[\cos(3t)-\cos(5t)\right]$$

3. (a) Express the solution of the given initial value problem in terms of a convolution integral:

$$y'' - 4y' + 20y = g(t), \quad y(0) = 1, y'(0) = 0. \qquad (1)$$

$$\mathcal{L}\{y'' - 4y' + 20y\} = \mathcal{L}\{g(t)\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - 4\big(sY(s) - y(0)\big) + 20Y(s) = G(s)$$

$$(s^2 - 4s + 20)Y(s) - s - 0 + 4 = G(s)$$

$$(s^2 - 4s + 20)Y(s) = G(s) + s - 4$$

$$Y(s) = \frac{G(s)}{s^2 - 4s + 20} + \frac{s-4}{s^2 - 4s + 20}$$

$$Y(s) = \frac{G(s)}{(s-2)^2 + 4^2} + \frac{s-4}{(s-2)^2 + 16}$$

$$Y(s) = G(s)\frac{1}{(s-2)^2 + 4^2} + \frac{s-2}{(s-2)^2 + 16} - \frac{1}{2}\cdot\frac{4}{(s-2)^2 + 16}$$

$$y(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{G(s)\frac{4}{(s-2)^2 + 4^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2 + 4^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2 + 4^2}\right\}$$

Use convolution Theorem for the first term

$$y(t) = \frac{1}{4}g(t) * \left(e^{2t}\sin 4t\right) + e^{2t}\cos 4t - \frac{1}{2}e^{2t}\sin 4t$$

$$y(t) = \frac{1}{4}\int_0^t g(\tau)\, e^{2(t-\tau)}\sin 4(t-\tau)\, d\tau + e^{2t}\left(\cos 4t - \frac{1}{2}\sin 4t\right)$$

(b) (**bonus-15 points**) Find the solution of the same initial value problem (1) using the method of variation of parameter. Show that your answer coincides with the answer obtained in item (a).

$$y'' - 4y' + 20y = g(t), \quad y(0) = 1, y'(0) = 0.$$

General solution: $y(t) = y_p(t) + y_h(t)$.

Find $y_h(t)$:

Characteristic equation.

$$r^2 - 4r + 20 = 0 \Rightarrow (r-2)^2 + 16 = 0$$
$$\Rightarrow r = 2 \pm 4i$$

$$y_h(t) = C_1 e^{2t}\cos 4t + C_2 e^{2t}\sin 4t$$

Variate parameters to find $y_p(t)$:

$$y_p(t) = u_1(t)\underbrace{e^{2t}\cos 4t}_{y_1} + u_2(t)\underbrace{e^{2t}\sin 4t}_{y_2}$$

Find Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} e^{2t}\cos 4t & e^{2t}\sin 4t \\ 2e^{2t}\cos 4t - 4e^{2t}\sin 4t & 2e^{2t}\sin 4t + 4e^{2t}\cos 4t \end{vmatrix}$$

$$= e^{4t}\left[\cos 4t\,(2\sin 4t + 4\cos 4t) - \sin 4t\,(2\cos 4t - 4\sin 4t)\right]$$

$$= e^{4t}\left[\cancel{2\cos 4t\sin 4t} + 4\cos^2 4t \cancel{- 2\sin 4t\cos 4t} + 4\sin^2 4t\right]$$

$$= 4e^{4t}(\cos^2 4t + \sin^2 4t) = \boxed{4e^{4t}}$$

$$u_1(t) = -\int_0^t \frac{g(\tau)\,y_2(\tau)}{W(y_1(\tau), y_2(\tau))}\,d\tau = -\int_0^t \frac{g(\tau)e^{2\tau}\sin 4\tau}{4e^{4\tau}}\,d\tau$$

$$= -\frac{1}{4}\int_0^t g(\tau)e^{-2\tau}\sin 4\tau\,d\tau$$

$$u_2(t) = \int_0^t \frac{g(\tau)\,y_1(\tau)}{W(y_1(\tau), y_2(\tau))}\,d\tau = \int_0^t \frac{g(\tau)e^{2\tau}\cos 4\tau}{4e^{4\tau}}\,d\tau$$

$$= \frac{1}{4}\int_0^t g(\tau)e^{-2\tau}\cos 4\tau\,d\tau$$

Then

$$y_p(t) = e^{2t}\cos 4t\left(-\frac{1}{4}\int_0^t g(\tau)e^{-2\tau}\sin 4\tau\,d\tau\right)$$
$$+ e^{2t}\sin(4t)\cdot\frac{1}{4}\int_0^t g(\tau)e^{-2\tau}\cos 4\tau\,d\tau$$

$$= \frac{1}{4}\int_0^t g(\tau)e^{2(t-\tau)}\left[\sin 4t\cos 4\tau - \cos 4t\sin 4\tau\right]d\tau$$

Using trigonometric identity

$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$$

we get

$$y_p(t) = \frac{1}{4}\int_0^t g(\tau)e^{2(t-\tau)}\sin 4(t-\tau)\,d\tau$$

see next page

General solution is

$$y(t) = \frac{1}{4}\int_0^t g(\tau)e^{2(t-\tau)}\sin 4(t-\tau)\,d\tau + C_1 e^{2t}\cos 4t + C_2 e^{4t}\sin 4t$$

It remains to find solution subject to the initial conditions:

$$y(0)=1\,,\quad y'(0)=0$$

We have:

$$y(0) = 0 + C_1 + 0 = 1 \Rightarrow C_1 = 1$$

$$y'(t) = \frac{1}{4}g(t)e^{2(t-t)}\sin 4(t-t) + 2C_1 e^{2t}\cos 4t$$
$$+ 4C_1 e^{2t}(-\sin 4t) + 2C_2 e^{2t}\sin 4t + 4C_2 e^{2t}\cos 4t$$

OR $y'(t) = e^{2t}(2C_1\cos 4t - 4C_1\sin 4t + 2C_2\sin 4t + 4C_2\cos 4t)$

Then

$$y'(0) = 2C_1 + 4C_2 = 0 \Rightarrow C_2 = -\frac{1}{2}C_1 = -\frac{1}{2}$$

Finally,

$$y(t) = \frac{1}{4}\int_0^t g(\tau)e^{2(t-\tau)}\sin 4(t-\tau)\,d\tau + e^{2t}\cos 4t - \frac{1}{2}e^{2t}\sin 4t$$

which coincides with the answer in item (a)