Homework Assignment #15 **Fall 2013 - MATH308**

due Wednesday Nov 13 at the beginning of class

Section covered 7.5

1. Given the following system of linear differential equations:

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 4x_1 + 3x_2 \end{cases} \qquad (1)$$

(a) Find the general solution of the system (1).

Coefficient matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

$tr(A) = 1+3 = 4$

$\det A = 3-8 = -5$

Characteristic polynomial

$\lambda^2 - (tr(A))\lambda + \det A = 0$

$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda-5)(\lambda+1) = 0$

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = -1$

Find eigenvectors v^1 and v^2, respectively, solving $(A - \lambda I)v = 0$

$\boxed{\lambda_1 = 5} \Rightarrow (A - 5I)v = 0 \Rightarrow \begin{pmatrix} 1-5 & 2 \\ 4 & 3-5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} -4v_1 + 2v_2 = 0 \\ 4v_1 - 2v_2 = 0 \end{matrix} \Rightarrow v_1 = \frac{v_2}{2}$

Set $v_2 = 2 \Rightarrow v_1 = 1$

Then $v^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{\lambda_2 = -1} \Rightarrow (A + I)v = 0 \Rightarrow \begin{pmatrix} 1+1 & 2 \\ 4 & 3+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \left.\begin{matrix} 2v_1 + 2v_2 = 0 \\ 4v_1 + 4v_2 = 0 \end{matrix}\right\} \Rightarrow v_1 = -v_2 \Rightarrow v^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Fundamental set $\{e^{\lambda_1 t} v^1, e^{\lambda_2 t} v^2\}$

$= \left\{ e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

General solution

$X(t) = C_1 e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) Find the solution of the the system (1) satisfying the initial conditions: $x_1(0) = 3, \quad x_2(0) = 0$.

By item (a): General solution

$$X(t) = C_1 e^{5t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$X(0) = C_1\begin{pmatrix}1\\2\end{pmatrix} + C_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} \Rightarrow \left.\begin{matrix}C_1 + C_2 = 3 \ (1)\\ 2C_1 - C_2 = 0 \ (2)\end{matrix}\right\} \Rightarrow$$

add $(2)+(1)$ $\Rightarrow$ $3C_1 = 3 \Rightarrow C_1 = 1 \Rightarrow C_2 = 3 - C_1 = 2$

$$X(t) = e^{5t}\begin{pmatrix}1\\2\end{pmatrix} + 2e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

(c) Find all α_1 and α_2 such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is the solution of of the system (1) with initial condition $x(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ then $x(t) \to 0$ as $t \to \infty$.

General solution

$$X(t) = C_1 e^{5t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\lim_{t\to+\infty} x(t) = 0 \iff C_1 = 0 \Rightarrow$$

$$x(t) = C_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow$$

$$x(0) = C_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} \Rightarrow \begin{matrix}\alpha_1 = C_2\\ \alpha_2 = -C_2\end{matrix} \Rightarrow$$

$$\boxed{\alpha_1 = -\alpha_2}$$

(d) Find all β_1 and β_2 such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is the solution of of the system (1) with initial condition $x(0) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ then $x(t) \to 0$ as $t \to -\infty$.

Similarly, $x(t) \xrightarrow[t\to-\infty]{} 0 \iff C_2 = 0 \Rightarrow$

$$x(t) = C_1 e^{5t}\begin{pmatrix}1\\2\end{pmatrix} \Rightarrow x(0) = C_1\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} \Rightarrow$$

$$\left\{\begin{matrix}\beta_1 = C_1\\ \beta_2 = 2C_1\end{matrix}\right. \Rightarrow \boxed{\beta_2 = 2\beta_1}$$

2. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= -x_1 + x_2 \\ x_2' &= x_1 + 2x_2 + x_3 \\ x_3' &= 3x_2 - x_3 \end{cases} \tag{2}$$

(a) Find the general solution of the system (2).

Coefficient matrix $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

Characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = -(1+\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -1-\lambda \end{vmatrix}$$

$$= -(1+\lambda)\left[(2-\lambda)(-1-\lambda) - 3\right] + (1+\lambda)$$

$$= -(1+\lambda)\left[-2 + \lambda^2 + \lambda - 2\lambda - 3\right] + 1 + \lambda$$

$$= -(1+\lambda)\left[\lambda^2 - \lambda - 5\right] + 1 + \lambda = -(1+\lambda)(\lambda^2 - \lambda - 5 - 1)$$

$$= -(1+\lambda)(\lambda^2 - \lambda - 6) = -(1+\lambda)(\lambda - 3)(\lambda + 2) = 0$$

Eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = -2$

Find the corresponding eigenvectors v^1, v^2, v^3

$\boxed{\lambda_1 = -1} \Rightarrow (A - \lambda_1 I)v = 0 \Rightarrow (A + I)v = 0$

$$\Rightarrow \begin{pmatrix} -1+1 & 1 & 0 \\ 1 & 2+1 & 1 \\ 0 & 3 & -1+1 \end{pmatrix} v = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v_2 = 0$

$v_1 + 3v_2 + v_3 = 0 \Rightarrow v_1 + v_3 = 0$

$3v_2 = 0$ set $v_3 = 1 \Rightarrow v_1 = -1$

$$\boxed{v^1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$\boxed{\lambda_2 = 3}$ $(A-\lambda_2 I)v = 0 \Rightarrow (A-3I)v = 0 \Rightarrow$

$$\begin{pmatrix} -1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 3 & -1-3 \end{pmatrix} v = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{pmatrix} v = 0$$

Augmentes matrix

$$\left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_2 + \frac{1}{4}R_1} \left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_3 + 4R_2} \left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-4v_1 + v_2 = 0$

$-\frac{3}{4}v_2 + v_3 = 0 \Rightarrow -3v_2 + 4v_3 = 0$

Set $v_1 = 1$

$\rightarrow v_2 = 4v_1 = 4 \rightarrow v_3 = \frac{3}{4}v_2 = 3$

$$\boxed{v^2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}}$$

$\boxed{\lambda_3 = -2}$ $(A - \lambda_3 I)v = 0 \Rightarrow (A + 2I)v = 0$

Augmented matrix

$$\left(\begin{array}{ccc|c} -1+2 & 1 & 0 & 0 \\ 1 & 2+2 & 1 & 0 \\ 0 & 3 & -1+2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_2 - R_1}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$v_1 + v_2 = 0$

$3v_2 + v_3 = 0$

Set $v_1 = 1$

$v_2 = -v_1 = -1$

$v_3 = -3v_2 = 3$

$\Rightarrow \boxed{v^3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}$

Finally, the general solution is

$$X(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

(b) Find the solution of the the system (2) satisfying the initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 14 \end{pmatrix}$

Using (a):

$$X(0) = c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 14 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 4 & -1 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 14 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 5 \\ 1 & 3 & 3 & 14 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3 + R_1} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 5 \\ 0 & 4 & 4 & 20 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_3 - R_2} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & 5 & 15 \end{array}\right)$$

$$-c_1 + c_2 + c_3 = 6$$

$$\left.\begin{array}{r} 4c_2 - c_3 = 5 \\ 5c_3 = 15 \Rightarrow \boxed{c_3 = 3} \end{array}\right\} c_2 = \frac{5 + c_3}{4} \Rightarrow \boxed{c_2 = 2}$$

$$c_1 = c_2 + c_3 - 6 = 2 + 3 - 6 \Rightarrow \boxed{c_1 = -1}$$

$$X(t) = -e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 2e^{3t}\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + 3e^{-2t}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$