Homework Assignment #16 Fall 2013

due Monday Nov 18 at the beginning of class

Section covered 7.6

1. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= 6x_1 - x_2 \\ x_2' &= 5x_1 + 2x_2 \end{cases}$$

(a) Find the general solution of the system (1).

$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$

$\operatorname{tr} A = 6+2 = 8$

$\det A = 12+5 = 17$

Characteristic equation

$\lambda^2 - 8\lambda + 17 = 0 \Rightarrow (\lambda-4)^2 + 1 = 0 \Rightarrow \lambda_{1,2} = 4 \pm i$

Eigenvector: $(A - (4-i)I)v = 0$

$$\left(\begin{array}{cc|c} 6-(4+i) & -1 & 0 \\ 5 & 2-(4+i) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -(2+i) & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow \frac{(2-i)}{5} R_2} \left(\begin{array}{cc|c} 2-i & -1 & 0 \\ 2-i & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_2 - R_1}$$

$$= \left(\begin{array}{cc|c} 2-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow (2-i)v_1 - v_2 = 0$$

Set $v_1 = 1 \Rightarrow v_2 = 2-i$

$v = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$$e^{\lambda t} v = e^{(4+i)t} \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$= e^{4t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$= \underbrace{e^{4t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}}_{Re(e^{\lambda t} v)} + i \underbrace{e^{4t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}}_{Im(e^{\lambda t} v)}$$

$$x(t) = c_1 e^{4t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

(b) If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a solution of (1), what is the limit of $x(t)$ as $t \to -\infty$. Does this limit depend on initial conditions?

$$x(t) = c_1 e^{4t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

$$\lim_{t \to -\infty} e^{4t} = 0 \Rightarrow \lim_{t \to -\infty} x(t) = 0$$ independently of c_1 & c_2

(c) Find the solution of the system (1) satisfying the initial conditions: $x_1(0) = -3, \quad x_2(0) = 2.$

$$x(t) = c_1 e^{4t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

$$x(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$c_1 = -3$$

$$2c_1 - c_2 = 2 \Rightarrow c_2 = 2c_1 - 2 = -6 - 2 = -8$$

$$x(t) = -3e^{4t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} - 8e^{4t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

2. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= 2x_1 + 5x_2 + x_3 \\ x_2' &= -5x_1 - 6x_2 + 4x_3 \\ x_3' &= 2x_3 \end{cases} \tag{2}$$

(a) It is known that $\begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} e^{2t}$ is a particular solution of the system and vector $\begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $-2+3i$ of the coefficient matrix. Find the (real) general solution of the system (2).

Fundamental set

$$\left\{ \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} e^{2t},\ \mathrm{Re}\left(e^{(-2+3i)t} \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} \right),\ \mathrm{Im}\left(e^{(-2+3i)t} \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} \right) \right\}$$

$$e^{(-2+3i)t} \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} = e^{-2t}(\cos 3t + i \sin 3t) \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} 4\cos 3t - 3\sin 3t + i(3\cos 3t + 4\sin 3t) \\ -5\cos 3t - i\,5\sin 3t \\ 0 \end{pmatrix}$$

$$\Rightarrow \mathrm{Re}\left(e^{(-2+3i)t} \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} \right) = e^{-2t} \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix}$$

$$\mathrm{Im}\left(e^{(-2+3i)t} \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} \right) = e^{-2t} \begin{pmatrix} 3\cos 3t + 4\sin 3t \\ -5\sin 3t \\ 0 \end{pmatrix}$$

General solution

$$X(t) = C_1 e^{2t} \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 3\cos 3t + 4\sin 3t \\ -5\sin 3t \\ 0 \end{pmatrix}$$

(b) Find the solution of the the system (2) satisfying the initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 25 \end{pmatrix}$

$$X(t) = C_1 e^{2t}\begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} + C_3 e^{-2t}\begin{pmatrix} 3\cos 3t + 4\sin 3t \\ -5\sin 3t \\ 0 \end{pmatrix}$$

$$X(0) = C_1\begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} + C_2\begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 25 \end{pmatrix}$$

$$\left.\begin{array}{l} 28C_1 + 4C_2 + 3C_3 = -2 \\ \left.\begin{array}{l} -5C_1 - 5C_2 = 3 \\ 25C_1 = 25 \Rightarrow C_1 = 1 \end{array}\right\} \Rightarrow C_2 = -\frac{8}{5} \end{array}\right\}$$

$$\Rightarrow 28 - \frac{32}{5} + 3C_3 = -2$$

$$3C_3 = -30 + \frac{32}{5} = -\frac{118}{15}$$

$$X(t) = e^{2t}\begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} - \frac{8}{5} e^{-2t}\begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} - \frac{118}{15} e^{-2t}\begin{pmatrix} 3\cos 3t + 4\sin 3t \\ -5\sin 3t \\ 0 \end{pmatrix}$$

(c) Find all $\alpha_1, \alpha_2, \alpha_3$ such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ is the solution of the system (2) with initial condition $x(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ then $x(t) \to 0$ as $t \to +\infty$.

$$x(t) = c_1 e^{2t} \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 3\cos 3t + 4\sin 3t \\ -5\sin 3t \\ 0 \end{pmatrix}$$

$\lim_{t \to +\infty} x(t) = 0$ if and only if $c_1 = 0$

$$x(0) = c_1 \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

It follows that $25 c_1 = \alpha_3$

$$c_1 = \frac{\alpha_3}{25} = 0$$

Conclusion $\alpha_3 = 0$

and arbitrary α_1, α_2