Homework assignment #17 Solutions MATH 308-511

Problem 1

Section 9.1 1-11 odd

Problem 1 a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

Eigenvalues $\det(A-\lambda I) = \lambda^2 - \mathrm{tr}A\,\lambda + \det A = \lambda^2 - \lambda - 6 + 4 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$

$\boxed{\lambda_1 = 2, \ \lambda_2 = -1}$

Eigenvector i) for $\lambda_1 = 2$: $(A-2I)v = 0 \iff \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_1 - 2v_2 = 0$

Set $v_2 = 1 \Rightarrow v_1 = 2 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda_1 = 2$

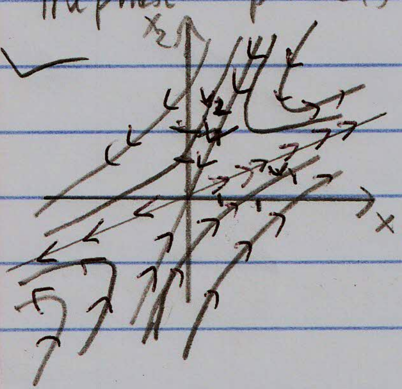
ii) for $\lambda_2 = -1$: $(A+I)v = 0 \iff \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 2v_1 - v_2 = 0$

Set $v_1 = 1 \Rightarrow v_2 = 2 \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvectors of $\lambda_2 = -1$

b) Since λ_1, λ_2 are real and $\lambda_1 > 0$, $\lambda_2 < 0$ the origin is a <u>saddle</u> and it is <u>unstable</u>.

The phase portrait is

c)



<u>Auxiliary</u> The general solution is

$$x(t) = \underbrace{c_1 e^{2t} v_1}_{\xi_1} + \underbrace{c_2 e^{-t} v_2}_{\xi_2}$$

$$\xi_1 = \frac{c}{\xi_2^2} \quad (\text{or } \xi_2 \equiv 0)$$

Problem 3 sec 9.1

a) $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

Eigenvalues: $\det(A-\lambda I) = \lambda^2 - \underbrace{trA}_{0}\lambda + \det A = \lambda^2 - 4 + 3 = \lambda^2 - 1$

$\Rightarrow \boxed{\lambda_1 = 1,\ \lambda_2 = -1}$

Eigenvectors: i) for $\lambda_1 = 1$ $\quad (A-I)v = 0 \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 - v_2 = 0$

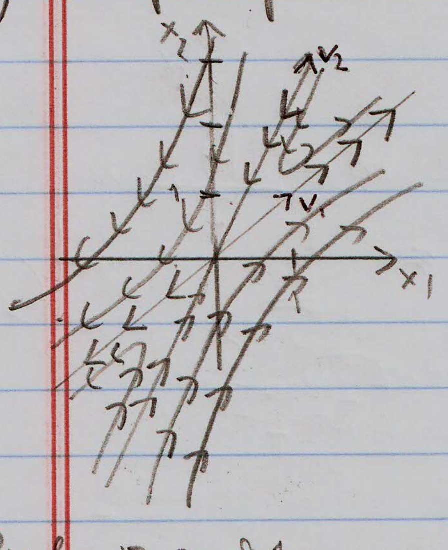
Set $v_2 = 1 \Rightarrow v_1 = 1 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

ii) for $\lambda_1 = -1$ $\quad (A+I)v = 0 \Leftrightarrow \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 3v_1 - v_2 = 0$

set $v_1 = 1 \Rightarrow v_2 = 3 \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of $\lambda_1 = -1$

b) Since λ_1 and λ_2 are real and of opposite signs, the origin is a <u>saddle point</u> and it is <u>unstable</u>

c) The phase portrait is



<u>Auxiliary</u> Gen. solution $x(t) = \underbrace{c_1 e^{t}}_{\xi_1} v_1 + \underbrace{c_2 e^{-t}}_{\xi_2} v_2$

$\Rightarrow \xi_2 = \frac{c}{\xi_1}$ (or $\xi_1 \equiv 0$)

Problem 5 sec 9.1

a) $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$

Eigenvalues: $\det(A-\lambda I) = \lambda^2 - trA\,\lambda + \det A = \lambda^2 + 2\lambda - 3 + 5 =$

$= \lambda^2 + 2\lambda + 2 = \lambda^2 + 2\lambda + 1 + 1 = (\lambda+1)^2 + 1 = 0 \Rightarrow \boxed{\lambda_{1,2} = -1 \pm i}$

Eigenvectors: i) for $\lambda_1 = -1+i$

$$(A-(-1+i)I)\,v = 0 \Leftrightarrow (A+(1-i)I)\,v = 0 \Leftrightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow$$

$v_1 + (-2-i)v_2 = 0$ (the first equation is obtained from the second by multiplication by $2-i$)

Set $v_2 = 1 \Rightarrow v_1 = 2+i \Rightarrow \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = -1+i$

ii) for $\lambda_2 = -1-i$. Since $\lambda_2 = \overline{\lambda_1}$ then as an eigenvector for λ_2 we can take $\overline{\begin{pmatrix} 2+i \\ 1 \end{pmatrix}} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

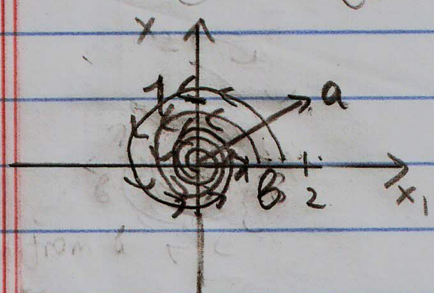
b) Since λ_1 and λ_2 are complex (non-real) and $\operatorname{Re}\lambda_1 < 0$ $(0,0)$ is <u>a spiral sink</u> and it is <u>asymptotically stable</u>

c) $V = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$

Auxiliary

$\alpha = -1,\ \beta = 1$

$a = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Gen. solution (as shown in class)

$$x(t) = e^{-t}R\cos(t-\delta)\,a - e^{-t}R\sin(t-\delta)\,b$$

for some R and δ

rotation from b to a

<u>Problem 7</u> a) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

<u>Eigenvalues</u> $\det(A-\lambda I) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 - 2\lambda - 3 + 8 =$

$= \lambda^2 - 2\lambda + 5 = (\lambda-1)^2 + 4 = 0 \Rightarrow \boxed{\lambda_{1,2} = 1 \pm 2i}$

(or $D = 4 - 20 = -16$

$\lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$)

Eigenvectors: i) for $\lambda_1 = 1+2i$

$$\left(A-(1+2i)I\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow (1-i)v_1 - v_2 = 0$$

(the second equation is obtained from the first by a multiplication by $1+i$). Set $v_1 = 1 \Rightarrow v_2 = 1-i \Rightarrow \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ is an eigenvector

for $\lambda_1 = 1+2i$

ii) for $\lambda_2 = 1-2i$, since $\lambda_2 = \overline{\lambda_1}$ an eigenvector

for λ_2 can be taken as $\overline{\begin{pmatrix} 1 \\ 1-i \end{pmatrix}} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

b) Since $\lambda_{1,2}$ are nonreal and $\mathrm{Re}\,\lambda_{1,2} > 0$

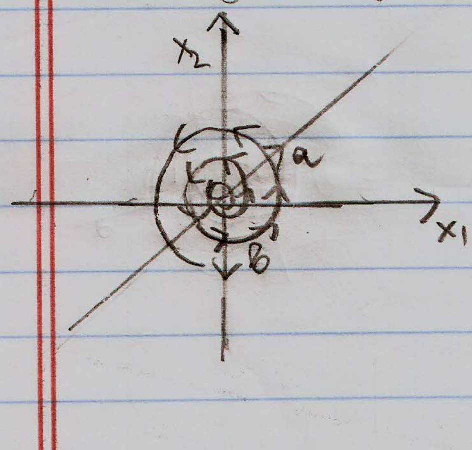
$(0,0)$ is a <u>spiral source</u> $\Rightarrow$ <u>unstable</u>

c) $v = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

<u>Auxiliary</u>

$\alpha = 1 \quad \beta = 2$

$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



Gen. solution is

$$x(t) = e^t R\cos(2t-\delta)\,a - e^t R\sin(2t-\delta)\,b$$

rotation from b to a

Problem 9 a) $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

Eigenvalues : $\det(A-\lambda I) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 - 2\lambda - 3 + 4 =$

$= \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0$

$\lambda_{1,2} = 1$

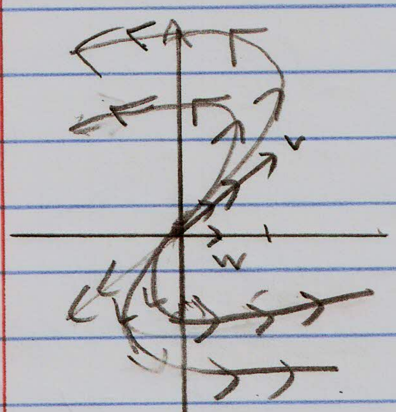
Eigenvectors: $(A-I)v = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$v_1 - 2v_2 = 0$

Set $v_2 = 1 \Rightarrow v_1 = 2 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 1$

b) Since $\lambda_1 = \lambda_2 > 0$ and geom. multiplicity < alg. multiplicity we have improper nodal source at the origin $\Rightarrow$ unstable

c)



Auxiliary.

Find the gen. eigenvector w s.t

$(A-I)w = v \Leftrightarrow$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow w_1 - 2w_2 = 1$

Set $w_2 = 0 \Rightarrow w_1 = 1 \Rightarrow w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Gen solution

$x(t) = c_1 e^t v + c_2 t e^t v + c_2 e^t w =$

$= \underbrace{(c_1 + c_2 t)e^t}_{\xi_1(t)} v + \underbrace{c_2 e^t}_{\xi_2(t)} w$

$\xi_1(t) = 0$ for exactly one t

$c_2 > 0 \Rightarrow \xi_2 > 0$ and $\xi_1(t) \xrightarrow[t\to+\infty]{} +\infty$

$c_2 < 0 \Rightarrow \xi_2 < 0$ and $\xi_1(t) \xrightarrow[t\to-\infty]{} -\infty$

All trajectories, when $t \to -\infty$ have tangent lines converging to v.

Problem 11 a) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$

$\lambda_1 = \lambda_2 = -1$ and any $v \neq 0$ is an eigenvector

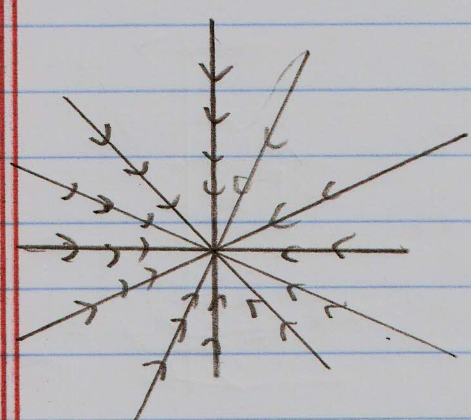
b) $\lambda_1 = \lambda_2 = -1$ and geom. multiplicity = alg. multiplicity

$\Rightarrow$ proper node. Since $\lambda_1 = \lambda_2 < 0$ it is asymptotically (proper nodal sink)

stable.

c) $$x(t) = \underbrace{c_1 e^{-t}}_{\xi_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{c_2 e^{-t}}_{\xi_2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\xi_2 = c\,\xi_1 \to$ straight lines



Problem 2 of the hwk

(a) $$A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{N}$$

$DN = ND$ ($D = \lambda I$ and it commutes with any 2×2 matrix)

$$e^D = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix}$$

$$e^N = I + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \dots$$

but note that $N^2 = NN = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 \Rightarrow N^k = 0$ for any $k \geq 2$

$\Rightarrow$

$$e^N = I + N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\Downarrow$

$$e^A = e^D e^N = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} e^\lambda & 0 \\ e^\lambda & e^\lambda \end{pmatrix}}$$

f) $$A = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}}_{D\text{-diagonal}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{N}$$

$$e^D = \begin{pmatrix} e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 \\ 0 & 0 & e^\lambda \end{pmatrix}$$

$$e^N = I + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \dots$$

but note that $N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

and $N^3 = N^2 N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0 \Rightarrow$

$N^k = 0$ for any $k \geq 3 \Rightarrow$

$$e^N = I + N + \frac{N^2}{\underset{2}{2!}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$e^{A} = e^{D} e^{N} = \boxed{\begin{pmatrix} e^{-1} & 0 & 0 \\ e^{-1} & e^{-1} & 0 \\ \frac{1}{2}e^{-1} & e^{-1} & e^{-1} \end{pmatrix}}$$

c) Note that if we denote by $A_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{pmatrix}$ then

$$e^{A} = \begin{pmatrix} e^{A_1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & e^{A_2} \end{pmatrix}$$

By analogy with item a) (or according to what we discussed in class

$$e^{A_1} = \begin{pmatrix} e^{\lambda_1} & e^{\lambda_1} \\ 0 & e^{\lambda_1} \end{pmatrix}, \quad e^{A_2} = \begin{pmatrix} e^{\lambda_2} & e^{\lambda_2} \\ 0 & e^{\lambda_2} \end{pmatrix} \Rightarrow$$

$$e^{A} = \boxed{\begin{pmatrix} e^{\lambda_1} & e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_1} & 0 & 0 \\ 0 & 0 & e^{\lambda_2} & e^{\lambda_2} \\ 0 & 0 & 0 & e^{\lambda_2} \end{pmatrix}}$$