Homework Assignment #17 **Fall 2013 - MATH308**

due Monday Nov. 25 at the beginning of class

Section covered: 7.5 (the case when there are repeated eigenvalues and a basis of eigenvectors)& 7.8 (when there are repeated eigenvalues but no basis of eigenvectors)

1. Find the general solution of the following system of linear differential equations:

$$\begin{cases} x_1' &= x_1 - 2x_2 + 2x_3 \\ x_2' &= -2x_1 + x_2 - 2x_3 \\ x_3' &= 2x_1 - 2x_2 + x_3 \end{cases}$$

if it is known that $\begin{pmatrix} e^{5t} \\ -e^{5t} \\ e^{5t} \end{pmatrix}$ is a particular solution of this system.

$\Rightarrow$ $\lambda_1 = 5$ is one of 3 possible eigenvalues of the coefficient matrix

and $v^1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is the corresponding eigenvector

Find the remaining eigenvalues and eigenvectors

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{pmatrix}$$

Characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)\underbrace{\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix}}_{\substack{\lambda^2 - 2\lambda - 3 \\ = (\lambda+1)(\lambda-3)}} + 2\underbrace{\begin{vmatrix} -2 & -2 \\ 2 & 1-\lambda \end{vmatrix}}_{\substack{2(\lambda-1)+4 \\ = 2\lambda - 2 + 4 = \\ = 2\lambda + 2 \\ = 2(\lambda+1)}} + 2\underbrace{\begin{vmatrix} -2 & 1-\lambda \\ 2 & -2 \end{vmatrix}}_{\substack{4 - 2 + 2\lambda \\ = 2 + 2\lambda \\ = 2(\lambda+1)}}$$

$$= (\lambda+1)\left[(1-\lambda)(\lambda-3) + 2\cdot 2 + 2\cdot 2\right]$$

$$= (\lambda+1)\left[\lambda - \lambda^2 - 3 + 3\lambda + 8\right] = (\lambda+1)(-\lambda^2 + 4\lambda + 5)$$

$$= -(\lambda+1)^2(\lambda - 5) = 0$$

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = -1$ (of multiplicity 2)

$\longrightarrow$

Find eigenvectors v^2 and v^3 corresponding to $\lambda_2 = -1$

$$A - \lambda_2 I | 0 = A + I | 0 = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow R_3 - R_1]{R_2 \leftrightarrow R_2 + R_1}$$

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow 2v_1 - 2v_2 + 2v_3 = 0$$

OR $v_1 - v_2 + v_3 = 0$

Set $v_3 = 0$, $v_2 = 1 \Rightarrow v_1 = 1 \Rightarrow v^2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Set $v_2 = 0$, $v_3 = 1 \Rightarrow v_1 = -1 \Rightarrow v^3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

two non. collinear vectors

Fundamental set

$$\left\{ e^{5t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

General solution

$$X(t) = C_1 e^{5t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

2. Find the general solution of the following system of linear differential equations:

$$\begin{cases} x_1' &= -x_1 + 3x_2 \\ x_2' &= -3x_1 + 5x_2 \end{cases} \tag{1}$$

Coefficient matrix $A = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix}$ $\quad tr A = 4$ $\quad \det A = 4$

Characteristic polynomial

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0$$

$\lambda = 2$ is eigenvalue of multiplicity 2

Eigenvectors: $(A - 2I)v = 0 \Rightarrow \left(\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$-v_1 + v_2 = 0 \Rightarrow v_2 = v_1 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Find a generalized eigenvector W such that

$$(A - 2I)W = V \Rightarrow \left(\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -3 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\boxed{-3W_1 + 3W_2 = 1}$$

Set $w_1 = 0 \Rightarrow w_2 = \frac{1}{3} \Rightarrow w = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$

General solution:

$$X(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\left(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}\right)$$

3. **(bonus 50 points)** Find the general solution of the following system of linear differential equations:

$$\begin{cases} x_1' &= -x_1 + x_2 + x_3 \\ x_2' &= -4x_1 - 6x_2 - 7x_3 \\ x_3' &= 3x_1 + 3x_2 + 4x_3 \end{cases}$$

if it is known that the characteristic polynomial is equal to $-\lambda^3 - 3\lambda^2 + 4$.

Solution: 1) Find the eigenvalues by solving

$$-\lambda^3 - 3\lambda^2 + 4 = 0 \iff \lambda^3 + 3\lambda^2 - 4 = 0$$

Guess: $\lambda = 1$ is a root $\Rightarrow \lambda^3 + 3\lambda^2 - 4$ is divisible by

$\lambda = 1$: $\quad \lambda - 1$

$$\begin{array}{l} \quad\ \lambda^2 + 4\lambda + 4 \\ \hline \lambda^3 + 3\lambda^2 \qquad - 4 \\ \lambda^3 - \lambda^2 \\ \hline \qquad 4\lambda^2 \\ \qquad 4\lambda^2 - 4\lambda \\ \hline \qquad\qquad 4\lambda - 4 \end{array}$$

$$\lambda^3 + 3\lambda^2 - 4 = (\lambda - 1)(\lambda^2 + 4\lambda + 4) = (\lambda - 1)(\lambda + 2)^2 \Rightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -2 \text{ (alg. mult. 2)} \end{array}$$

2) Find the eigenspace of $\lambda = -2$:

$$(A+2I)v = 0 \iff \begin{pmatrix} 1 & 1 & 1 \\ -4 & -4 & -7 \\ 3 & 3 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -4 & -4 & -7 & 0 \\ 3 & 3 & 6 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 4R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_2]{R_2 \to -\frac{1}{3}R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} v_1 + v_2 + v_3 = 0 \\ v_3 = 0 \end{array} \Rightarrow v_1 = -v_2 \Rightarrow \text{Set } v_2 = 1 \Rightarrow v_1 = -1$$

The eigenspace is one-dim. and consists of all multiples of $v = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Consequently, the geom. multiplicity of $\lambda=-2$ is equal to $1 <$ alg. multiplicity of $(\lambda=-2)$

Find the **generalized eigenvalue** by solving

$$(A+2I)w = \underbrace{v}_{\text{found in the previous step}}$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -4 & -4 & -7 \\ 3 & 3 & 6 \end{pmatrix} w = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ -4 & -4 & -7 & 1 \\ 3 & 3 & 6 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 4R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_2]{R_2 \to -\frac{1}{3} R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} w_1 + w_2 + w_3 = -1 \\ w_3 = 1 \\ w_1 + w_2 = -2 \end{array}$$

Take $w_2 = -2 \Rightarrow w_1 = 0$

$$\Rightarrow \quad w = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

3) Find the eigenline, corresponding to the non-repeated eigenvalue $\lambda = 1$

$$(A - I)z = 0 \Leftrightarrow \begin{pmatrix} -2 & 1 & 1 \\ -4 & -7 & -7 \\ 3 & 3 & 3 \end{pmatrix} z = 0 \Rightarrow$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ -4 & -7 & -7 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right) \underset{R_3 \to 2R_3 + 3R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -11 & -11 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right) \underset{R_3 \to R_3 + \frac{5}{11}R_2}{\overset{R_2 \to -\frac{1}{11}R_2}{\sim}} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad \begin{aligned} -2z_1 + z_2 + z_3 &= 0 \\ z_2 + z_3 &= 0 \end{aligned} \quad \Rightarrow z_1 = 0;$$ setting $z_3 = 1$ we get $z_2 = -1 \Rightarrow z = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

In the basis (v, w, z)

$$Av = -2v$$
$$Aw = v - 2w \qquad \left(\text{because } (A+2I)w = z \right)$$
$$Az = z$$

$$\Rightarrow A\,(v\ w\ z) = (v\ w\ z)\underbrace{\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{B} \Rightarrow$$

$(v\ w\ z)\, e^{Bt}$ is the fundamental matrix

$$B = \begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ 0 & -2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{1} \end{pmatrix} \begin{matrix} \to B_1 \\ \to B_2 \end{matrix} \to \text{block-diagonal} \qquad e^{Bt} = \begin{pmatrix} e^{B_1 t} & 0 \\ 0 \quad 0 & e^{t} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \Longrightarrow$$

$$(v \; w \; z)\, e^{Bt} = \Big(e^{-2t}\underbrace{v}_{\begin{pmatrix}-1\\1\\0\end{pmatrix}},\; te^{-2t}\underbrace{v}_{\begin{pmatrix}-1\\1\\0\end{pmatrix}} + e^{-2t}\underbrace{w}_{\begin{pmatrix}0\\-2\\1\end{pmatrix}},\; e^{t}\underbrace{z}_{\begin{pmatrix}0\\-1\\1\end{pmatrix}} \Big)$$

$\Rightarrow$ the general solution

$$x(t) = c_1 e^{-2t}\begin{pmatrix}-1\\1\\0\end{pmatrix} + c_2\left(te^{-2t}\begin{pmatrix}-1\\1\\0\end{pmatrix} + e^{-2t}\begin{pmatrix}0\\-2\\1\end{pmatrix}\right) + c_3 e^{t}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$