MAT4 328
Fall 2013

Homework #1 Solutions

Problem 1 (a) If $v(t) \equiv V_e$ is a solution then

$$0 = v'(t) = 9.8 - \frac{v(t)}{30} = 9.8 - \frac{V_e}{30} \Rightarrow$$

$$9.8 - \frac{V_e}{30} = 0 \Rightarrow V_e = 30 \times 9.8 = \boxed{294}$$

(b) As derived in class the equation

$v' = av + b$ has a general solution

$$v(t) = -\frac{b}{a} + Ce^{at}$$

In our case $b = 9.8$, $a = -\frac{1}{30} \Rightarrow -\frac{b}{a} = 294 (= V_e)$

The general solution of our equation is

$$v(t) = 294 + Ce^{-\frac{t}{30}}$$

Rem You can also derive this formula directly using separation of variables, see for example solution of Problem 1(b) of Homework #1 of Spring 201 posted on www.math.tamu.edu/~zelenko/ODEhw

Determine C from the initial conditions $v(0) = 98$

$$98 = 294 + Ce^0 = 294 + C \Rightarrow C = 98 - 294 = -196$$

$\rightarrow v(t) = \boxed{294 - 196e^{-\frac{t}{30}} = 98\left(3 - 2e^{-\frac{t}{30}}\right)}$

$\lim_{t \to +\infty} v(t) = \lim_{t \to +\infty} 294 - 196e^{-t/30} =$

$= 294 - 196 \underbrace{\lim_{t \to \infty} e^{-t/30}}_{0} = \boxed{294 = v_e}$ i.e.

The limiting velocity equal to the equilibrium velocity found in (a)

(c) We have to solve the equation

$294 - 196e^{-t/30} = \frac{2}{3} \cdot 294 = 196 \Leftrightarrow$

$\cancel{98}\left(3 - 2e^{-t/30}\right) = 2 \cdot \cancel{98} \Leftrightarrow 3 - 2e^{-t/30} = 2 \Rightarrow$

$1 - 2e^{-t/30} = 0 \Leftrightarrow e^{-t/30} = \frac{1}{2} \Leftrightarrow e^{t/30} = 2 \Rightarrow$

$\frac{t}{30} = \ln 2 \Leftrightarrow \boxed{t = 30\ln 2}$

(d) Let $s(t)$ be the distance the object has fallen in time $t \Rightarrow s'(t) = v(t)$, $s(0) = 0 \Rightarrow$

$s(30\ln 2) = \int_0^{30\ln 2} v(t)\,dt = \int_0^{30\ln 2} 98\left(3 - 2e^{-t/30}\right)dt =$

$$= 98\left(3t + 60e^{-t/30}\right)\Big|_0^{30\ln 2} = 98\Big(90\ln 2 + 60\underbrace{e^{-\ln 2}}_{\frac{1}{2}} -$$

$$-60\Big) = 98(90\ln 2 + 30 - 60) = \boxed{98(90\ln 2 - 30)} =$$

$$= \boxed{8820\ln 2 - 2940}$$

Problem 2

(a) $(1+x^2)^{1/3}y' + xy^2 = 0 \Leftrightarrow (1+x^2)^{1/3}y' = -xy^2 \Leftrightarrow$ (separate)

$$\frac{dy}{y^2} = -\frac{x\,dx}{(1+x^2)^{1/3}} \Leftrightarrow \int\frac{dy}{y^2} = -\int\frac{x\,dx}{(1+x^2)^{1/3}} \Leftrightarrow$$

u-substitution: $u = 1+x^2$, $du = 2x\,dx$

$$\Leftrightarrow -\frac{1}{y} = -\frac{1}{2}\int u^{-1/3}du =$$

$$= -\frac{1}{2}\,\frac{u^{2/3}}{\frac{2}{3}} + C = -\frac{3}{4}(1+x^2)^{2/3} + C \Rightarrow$$

$$\frac{1}{y} = \frac{3}{4}(1+x^2)^{2/3} - C \Rightarrow \boxed{y = \frac{1}{\frac{3}{4}(1+x^2)^{\frac{2}{3}} - C}}$$

(b) $dx + x^4 \sin y\,dy = 0 \Leftrightarrow x^4\sin y\,dy = -dx \Leftrightarrow$

$\sin y\,dy = -x^{-4}dx \Leftrightarrow \int \sin y\,dy = -\int x^{-4}dx \Leftrightarrow$

$-\cos y = \cancel{-}\dfrac{x^{-3}}{\cancel{-}3} + C \Leftrightarrow \boxed{\cos y = -\frac{1}{3}x^{-3} - C}$