

# MATH308/504 Homework assignment 1 Solution

1) (a)  $v_e$  must satisfy the following equation

$$19.6 - \frac{v_e}{10} = 0 \Rightarrow \boxed{v_e = 196}$$

(b)  $v' = 19.6 - \frac{v}{10}$

W. r. t.  $\frac{v'}{19.6 - \frac{v}{10}} = 1$

Integrate w.r.t.  $\int \frac{v' dt}{19.6 - \frac{v}{10}} = \int dt \Rightarrow$

$$\int \frac{dv dt}{19.6 - \frac{v}{10}} = t + C_1 \Rightarrow$$

$u = 19.6 - \frac{v}{10}$   
 $du = -\frac{dv}{10}$

$-10 \int \frac{du}{u} = t + C_1 \Rightarrow$

$$-10 \ln |u| = t + C_1 \Rightarrow \ln |19.6 - \frac{v}{10}| = -\frac{t}{10} + C_2 \Rightarrow$$

$$|19.6 - \frac{v}{10}| = \underbrace{e^{C_2}}_{C_3 > 0} e^{-\frac{t}{10}} \Rightarrow 19.6 - \frac{v}{10} = C_3 e^{-t/10}, C_3 \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \frac{v}{10} = 196 - C_3 e^{-t/10} \Rightarrow \text{the general solution is}$$

$$v(t) = 196 - C e^{-t/10}$$

If  $v(0) = 0$  then  $0 = 196 - C \Rightarrow C = 196 \Rightarrow$

The solution of the initial value problem is

$$\boxed{v(t) = 196 (1 - e^{-t/10})}$$

$$\boxed{\lim_{t \rightarrow +\infty} v(t) = 196 = v_e}$$

(c) We have to solve the equation

$$196(1 - e^{-t/10}) = \frac{1}{2} 196 \Leftrightarrow$$

$$1 - e^{-t/10} = \frac{1}{2} \Leftrightarrow e^{-t/10} = \frac{1}{2} \Rightarrow$$

$$-\frac{t}{10} = \ln \frac{1}{2} = -\ln 2 \Rightarrow \frac{t}{10} = \ln 2 \Rightarrow$$

$$\boxed{t = 10 \ln 2}$$

(d) Let  $s(t)$  be the distance the object has fallen in

time  $t$ .  $\Rightarrow s'(t) = v(t) \Rightarrow$

$$s(10 \ln 2) = \int_0^{10 \ln 2} v(t) dt = \int_0^{10 \ln 2} 196(1 - e^{-t/10}) dt =$$

$$= 196 \left( t + 10e^{-t/10} \right) \Big|_0^{10 \ln 2} = 196 \left( 10 \ln 2 + 10e^{-\ln 2} - 10 \cdot \frac{1}{2} \right)$$

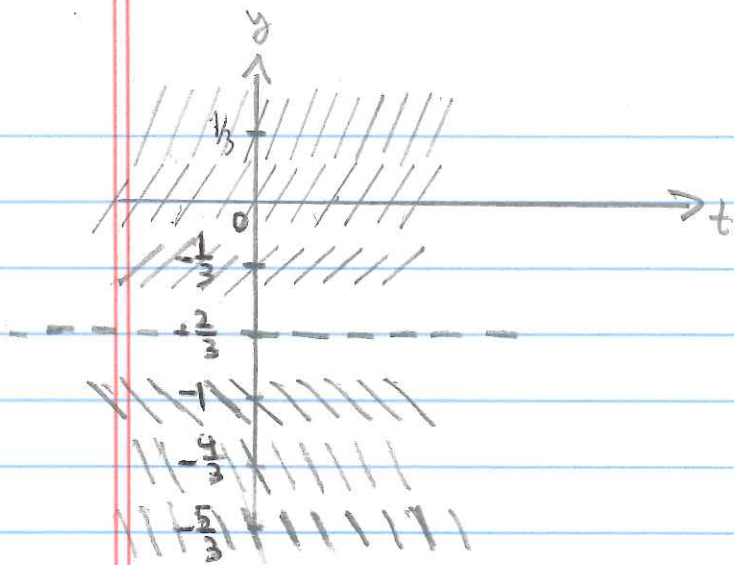
$$= 196 (10 \ln 2 - 5)$$

### Problem 2

(a) Equilibrium point:  $2 + 3y = 0 \Rightarrow y = -\frac{2}{3} \Rightarrow$   
equilibrium solution is  $y(t) = -\frac{2}{3}$

(b)  $y' = 2 + 3y$

$y$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-1$	$-\frac{2}{3}$	$-\frac{1}{3}$	$0$	$\frac{1}{3}$
slope	$-3$	$-2$	$-1$	$0$	$1$	$2$	$3$



$y' = 2 + 3y$   
 $y = -\frac{2}{3}$   
 $\lim_{t \rightarrow \infty} y(t) = \infty$   
 $\lim_{t \rightarrow \infty} y(t) = -\infty$

(c) If  $y_0 > -\frac{2}{3}$   $\lim_{t \rightarrow +\infty} y(t) = +\infty$

If  $y_0 < -\frac{2}{3}$   $\lim_{t \rightarrow +\infty} y(t) = -\infty$

If  $y \neq -\frac{2}{3}$   $\lim_{t \rightarrow -\infty} y(t) = -\frac{2}{3}$

(d)  $y' = 2 + 3y \Leftrightarrow \frac{y'}{2+3y} = 1 \Leftrightarrow \frac{dy}{2+3y} = dt \Leftrightarrow$

$$\int \frac{dy}{2+3y} = t + C_1$$

$$\frac{1}{3} \ln |2+3y| = t + C_1 \Leftrightarrow \ln |2+3y| = 3t + C_2 \Leftrightarrow 2+3y = C e^{3t}$$

$$y(t) = -\frac{2}{3} + C e^{3t}$$

If  $y(0) = y_0$  then  $y_0 = -\frac{2}{3} + C \Rightarrow C = y_0 + \frac{2}{3} \Rightarrow$

$$\boxed{y(t) = -\frac{2}{3} + \left(y_0 + \frac{2}{3}\right) e^{3t}}$$

From this explicit form

$$\lim_{t \rightarrow +\infty} y(t) = \begin{cases} +\infty & \text{if } y_0 + \frac{2}{3} > 0 \Leftrightarrow y_0 > -\frac{2}{3} \\ -\infty & \text{if } y_0 + \frac{2}{3} < 0 \Leftrightarrow y_0 < -\frac{2}{3} \end{cases}$$

$$\lim_{t \rightarrow -\infty} y(t) = -\frac{2}{3} \quad \text{as in item (c)}$$

### Problem 3

(a)  $(1-x^2)^{1/2} y' + xy = 0$

$$y' = -\frac{xy}{(1-x^2)^{1/2}} \rightarrow \text{separable equation}$$

$$\frac{dy}{y} = -\frac{x dx}{(1-x^2)^{1/2}} \Leftrightarrow \int \frac{dy}{y} = -\int \frac{x dx}{(1-x^2)^{1/2}} \Leftrightarrow$$

$$\ln|y| = -\int \frac{x dx}{(1-x^2)^{1/2}} + C_1$$

$$\boxed{y = C e^{-\int \frac{x dx}{(1-x^2)^{1/2}}}}$$

$$\begin{aligned} u &= 1-x^2 & \frac{du}{dx} &= -2x \\ \int \frac{x dx}{(1-x^2)^{1/2}} &= -\frac{1}{2} \int u^{-1/2} du = \\ &= -\frac{1}{2} \cdot (-2) u^{1/2} = u^{1/2} = (1-x^2)^{1/2} \end{aligned}$$

(b)  $y' + \frac{x \cos x}{y \cos y} = 0 \Leftrightarrow y' = -\frac{x \cos x}{y \cos y} \Leftrightarrow$

separable equation

$$\Leftrightarrow y \cos y dy = -x \cos x dx$$

$$\int y \cos y dy = -\int x \cos x dx + C$$

$$\boxed{y \sin y + \cos y = -x \sin x - \cos x + C}$$

$$\begin{aligned} \int x \cos x dx &= \\ &\downarrow \text{By part} \\ x \sin x - \int \sin x dx &= \\ &= x \sin x + \cos x \end{aligned}$$

### Problem 4

(a) Equilibrium point:  $y - y^2 = 0 \Leftrightarrow$

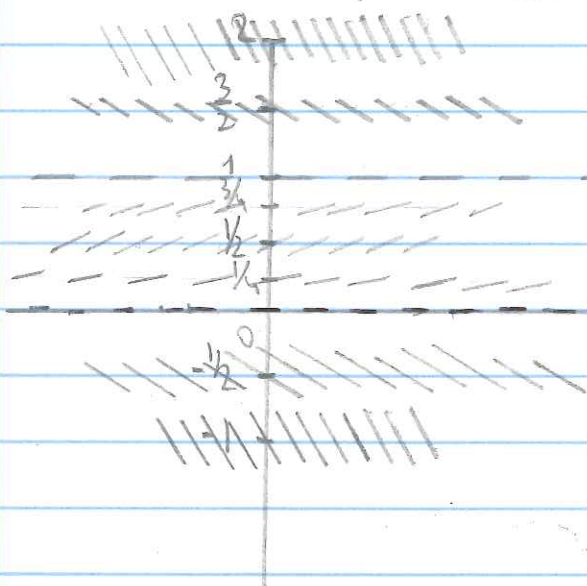
$$y(1-y) = 0 \Rightarrow y = 0 \text{ or } y = 1$$

(b)  $y(1-y) > 0$  

Slope is positive if  $0 < y < 1$

Slope is negative if  $y < 0$  or  $y > 1$

y	-1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2
slope	-2	$-\frac{3}{4}$	0	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	0	$-\frac{3}{4}$	-2



(c) i If  $y(0) = \frac{1}{2}$  since  $0 < y(0) < 1$  then by uniqueness

$0 < y(t) < 1$  for any  $t$  (here  $f(y) = y - y^2$   $\frac{df}{dy} = 1 - 2y$  is continuous  $\Rightarrow$

uniqueness holds)  $\Rightarrow y'(t)$  is positive for any  $t$  (the slope

is positive on the strip  $0 < y < 1$ )  $\Rightarrow y(t)$  is monotonically increasing  $\Rightarrow \lim_{t \rightarrow +\infty} y(t)$  and  $\lim_{t \rightarrow -\infty} y(t)$  exist and since it must be equal to an equilibrium point then

$$\lim_{t \rightarrow +\infty} y(t) = 1$$

$$\lim_{t \rightarrow -\infty} y(t) = 0$$

(c) (i) First, the same arguments will work for all  $0 < y_0 < 1$

Second, if  $y_0 \geq 1 \Rightarrow y(0) \geq 1 \Rightarrow$  by uniqueness  $y(t) \geq 1$

for any  $t \Rightarrow y'(t) < 0$  because the slope is negative

in the strip  $y > 1 \Rightarrow y(t)$  is monotonically decreasing &

$y(t) \geq 1 \Rightarrow \lim_{t \rightarrow +\infty} y(t)$  exists and since the limit must be equal to an equilibrium point  $\lim_{t \rightarrow +\infty} y(t) = 1$  again

Third, if  $y_0 < 0$  then by similar arguments  $y(t)$  is monotonically decreasing  $\Rightarrow y(t)$  does not go to 1 when  $t \rightarrow +\infty$

Conclusion  $y(t) \xrightarrow{t \rightarrow +\infty} 1 \Leftrightarrow \boxed{y_0 > 0}$

(c) (iii) As mentioned in the previous item  $y(t)$  is monotonically decreasing. Also its slopes go to  $-\infty \Rightarrow$   
 $\boxed{y(t) \text{ approach the value } -\infty \text{ as } t \text{ increases}}$

(d) bonus  $y' = y - y^2 \Leftrightarrow \frac{dy}{y - y^2} = dt \Leftrightarrow$

$$\int \frac{dy}{y - y^2} = \int dt$$

$$\frac{1}{y - y^2} = \frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y} \Rightarrow \int \frac{dy}{y - y^2} = \ln |y| - \ln |1 - y| = t + C_1$$

$$\ln \left| \frac{y}{1 - y} \right| = t + C_1 \Leftrightarrow$$

$$\frac{y}{1-y} = Ce^t$$

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Initial condition:  $y(0) = -1 \Rightarrow$

$$-\frac{1}{2} = C \Rightarrow C = -\frac{1}{2} \Rightarrow$$

$$\frac{y}{1-y} = -\frac{1}{2}e^t \Rightarrow y = \frac{1}{2}e^t y - \frac{1}{2}e^t \Rightarrow$$

$$y(t) = \frac{\frac{1}{2}e^t}{\frac{1}{2}e^t - 1} = \boxed{\frac{e^t}{e^t - 2}}$$

$$e^t \neq 2 \Rightarrow t \neq \ln 2. \quad \lim_{t \rightarrow \ln 2^-} y(t) = -\infty$$

Since the initial time is  $t_0 = 0$ , then the solution is defined for  $\boxed{t < \ln 2}$ . Approaching to  $\ln 2$  from the left  $\boxed{\text{it explodes to } -\infty}$ .