Homework assignment #2 Solutions MATH 308-505
Fall 2013

**Problem 1**

$$(xy^2+x)y' + x^2y - y = 0 \iff$$

$$\underbrace{(xy^2+x)}_{x(y^2+1)}y' = \underbrace{y - x^2y}_{y(1-x^2)} \iff \frac{y^2+1}{y}dy = \frac{1-x^2}{x}dx \iff$$

$$\left(\iff y' = \frac{1-x^2}{x}\cdot\frac{y}{y^2+1} \Rightarrow \underline{\text{separable}}\right)$$

$$\int\left(y+\frac{1}{y}\right)dy = \int\left(\frac{1}{x}-x\right)dx + C$$

$$\frac{y^2}{2} + \ln|y| = \ln|x| - \frac{x^2}{2} + C$$ – the general solution in implicit form.

The initial conditions $y(1)=1 \Rightarrow \frac{1}{2} + \underbrace{\ln 1}_{0} = \underbrace{\ln 1}_{0} - \frac{1}{2} + C \Rightarrow$

$$C = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow C = 1 \Rightarrow$$

$$\boxed{\frac{y^2}{2} + \ln|y| = \ln|x| - \frac{x^2}{2} + 1} \iff \boxed{\frac{x^2+y^2}{2} + \ln\left|\frac{y}{x}\right| = 1}$$

<u>Rem</u>: absolute values can be removed here because $x_0 > 0$ and $y(x_0) > 0$.

**Problem 2** $y' = y^2 + 5y + 6$

(a) Equilibrium points are solutions of the equation

$$y^2+5y+6 = 0 \iff (y+2)(y+3) = 0 \Rightarrow \underline{y = -2 \text{ or } y = -3}$$

Equivalently, we can use the quadratic formula

$$D = 5^2 - 4\cdot 6 = 25 - 24 = 1$$

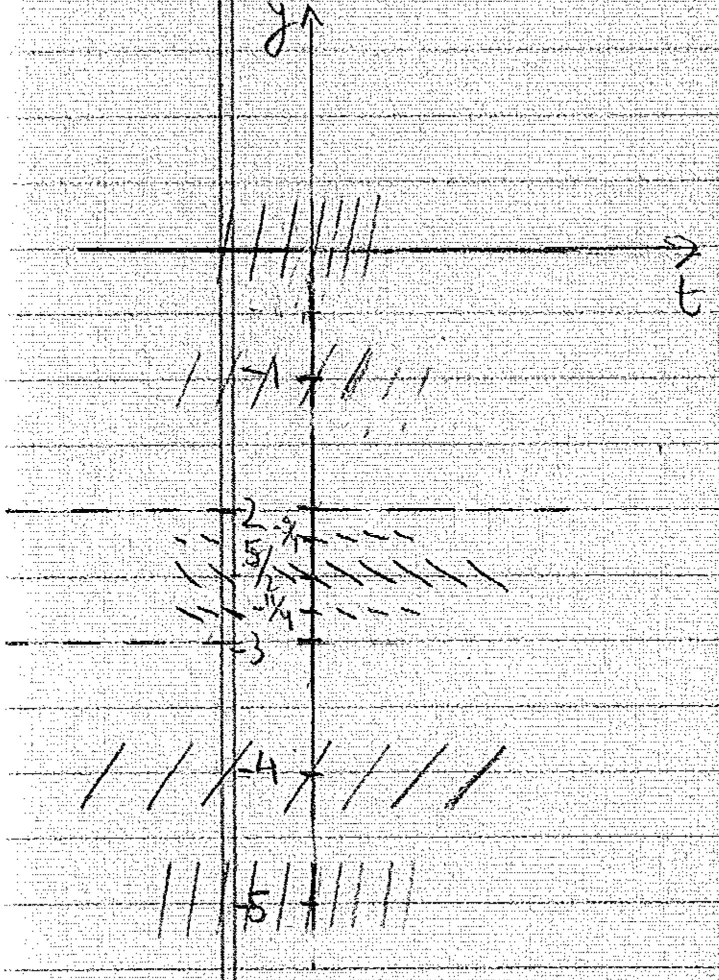
$$y_1 = \frac{-5+1}{2} = -2$$

$$y_2 = \frac{-5-1}{2} = -3$$

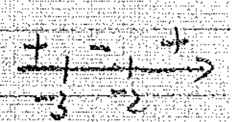
## b) The direction field

| $y$ | $-5$ | $-4$ | $-3$ | $-\frac{11}{4}$ | $-\frac{5}{2}$ | $-\frac{9}{4}$ |
|---|---|---|---|---|---|---|
| slope<br>$y^2+5y+6$ | $25-25+6$<br>$=\boxed{6}$ | $16-20+6=$<br>$=\boxed{2}$ | $0$ | $\frac{121}{16}-\frac{55}{4}+6=\frac{121-220+96}{16}=$<br>$=\boxed{-\frac{3}{16}}$ | $\frac{25}{4}-\frac{25}{2}+6=$<br>$=\boxed{-\frac{1}{4}}$ | $\frac{81}{16}-\frac{45}{4}+6=$<br>$=\frac{81-180+96}{16}=$<br>$=\boxed{-\frac{3}{16}}$ |

| $y$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|
| slope | $0$ | $1-5+6=$<br>$=2$ | $6$ |



Since $y^2+5y+6=(y+2)(y+3)$



the slope is positive if $y>-2$ or $y<-3$ and negative if $-3<y<-2$

c(i) If $y(0) = -\frac{5}{2}$, then from the existence and uniqueness theorem $-3 < y(t) < -2$ for any $t$

(because $y_0(t) \equiv -2$ and $y_2(t) \equiv -3$ are solutions and the graphs of two different solutions do not intersect)

$\Rightarrow y'(t) < 0$ (because the slopes are negative for $-3 < y < -2$) $\Rightarrow$ $y(t)$ is monotone decreasing and bounded below by $-3$ and above by $-2$. Therefore the limits $\lim_{t\to+\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$ exist. Also these limits must be equilibrium points $\Rightarrow$

$$\boxed{\begin{array}{l}\lim_{t\to+\infty} y(t) = -3 \\ \lim_{t\to-\infty} y(t) = -2\end{array}}$$

c (ii) Similar analysis shows that $\lim_{t\to+\infty} y(t) = -3$ for all $y_0 < -2$

c(iii) Since $y(0) = 0 > -2$ $y'(t) > 0 \Rightarrow y(t)$ is monotonically increasing. Besides $y^2 + 5y + 6 > 6$, if $y > 0$
$\Rightarrow$ $y(t)$ increases with the velocity $\geq 6$ at any $t > 0 \Rightarrow$
$y(t)$ will go to $+\infty$ as $t$ increases (actually

it will approach $+\infty$ in finite time as we will see in the next item)

d) $y' = y^2+5y+6 \Leftrightarrow$

$$\frac{dy}{y^2+5y+6} = dt \Rightarrow \int \frac{dy}{y^2+5y+6} = t + C_1$$

$$\frac{1}{y^2+5y+6} = \frac{1}{(y+2)(y+3)} \underset{1}{\equiv} \frac{A}{y+2} + \frac{B}{y+3} \Leftrightarrow$$

the method of undetermined coefficients

$$1 = A(y+3) + B(y+2)$$

To find A put $y=-2$: $\Rightarrow 1 = A$

To find B put $y=-3$: $\Rightarrow 1 = -B \Rightarrow B = -1$

$$\frac{1}{y^2+5y+6} = \frac{1}{y+2} - \frac{1}{y+3} \Leftrightarrow \int \frac{dy}{y^2+5y+6} = \int \frac{dy}{y+2} - \int \frac{dy}{y+3} =$$

$$= \ln\left|\frac{y+2}{y+3}\right| = t + C_1 \Rightarrow \left|\frac{y+2}{y+3}\right| = e^{t+C_1} = \underset{\overset{\|}{C}}{\boxed{e^{C_1}}}\, e^{t} \Rightarrow$$

$$\frac{y+2}{y+3} = Ce^{t}$$

$$y(0) = 0 \Rightarrow \frac{2}{3} = Ce^{0} \Rightarrow C = \frac{2}{3} \Rightarrow \frac{y+2}{y+3} = \frac{2}{3}e^{t}$$

$$(y+2) = \frac{2}{3}e^{t}(y+3) \Rightarrow \left(1 - \frac{2}{3}e^{t}\right)y = 2e^{t} - 2 \Rightarrow$$

$$\boxed{y(t) = \frac{2e^t - 2}{1 - \frac{2}{3}e^t} = \frac{6e^t - 6}{3 - 2e^t}};\ 1 - \frac{2}{3}e^t = 0 \Rightarrow e^t = \frac{3}{2} \Rightarrow t = \ln\frac{3}{2}$$

Note that the solution approach $+\infty$ as $t$ approach $\ln\frac{3}{2}$

$\Rightarrow$ the solution is defined for all $\boxed{t < \ln\frac{3}{2}}$

Problem 3

(a) $(y^2 - 2xy)dx + x^2 dy = 0 \Rightarrow$

$$\frac{dy}{dx} = -\frac{y^2 - 2xy}{x^2} = -\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) \quad (*)$$

Substitute $u = \frac{y}{x} \Rightarrow y = xu \Rightarrow$

$$\frac{dy}{dx} = \frac{d}{dx}(xu) = xu' + u \Rightarrow$$ Substituting into $(*)$

we get

$$xu' + u = -u^2 + 2u \Rightarrow$$

$$xu' = -u^2 + u \Rightarrow$$ separable

$$\frac{du}{u - u^2} = \frac{dx}{x} \Rightarrow \int \frac{du}{u - u^2} = \int \frac{dx}{x} + C \Rightarrow \int \frac{du}{u-1} - \int \frac{du}{u} = \ln|x| + C_1 \Rightarrow$$

$$\frac{1}{u - u^2} = \frac{1}{u-1} - \frac{1}{u}$$

$$\Rightarrow \ln|u-1| - \ln|u| = \ln|x| + C_1 \Rightarrow \ln\left|\frac{u-1}{u}\right| = \ln|x| + C_1 \Rightarrow$$

$$\left|\frac{u-1}{u}\right| = \underbrace{e^{C_1}}_{C>0}|x| \Rightarrow \frac{u-1}{u} = \underset{\text{any } C}{C}x \Rightarrow u - 1 = Cxu \Rightarrow$$

$$u(1 - Cx) = 1 \Rightarrow$$

(a) $u = \frac{1}{1-cx} \Rightarrow y = ux = \frac{x}{1-cx}$

(b) $xy' = y - xe^{y/x} \Rightarrow y' = \frac{y}{x} - e^{y/x}$

Substitude $u = \frac{y}{x} \Rightarrow y = xu \Rightarrow$

$$\frac{dy}{dx} = xu' + u = u - e^{u} \Rightarrow$$

$$xu' = -e^{u} \Rightarrow$$

$$-e^{-u}du = \frac{dx}{x} \Rightarrow$$

$$\int -e^{-u}du = \ln|x| + C_1 \Rightarrow$$

$$e^{-u} = \ln|x| + C \Rightarrow$$

$$-u = \ln(\ln|x|) + C \Rightarrow u = -(\ln(\ln|x|) + C) \Rightarrow$$

$$\Downarrow$$

$$y = xu = -x(\ln(\ln|x|) + C)$$