Homework # 3 MATH 308 - 505, FALL 2013
Solutions

Problem 1

$$t^2 y' + 3ty = \cos 2t \Rightarrow$$

$$y' + \frac{3}{t} y = \frac{\cos 2t}{t^2}$$

$$\Downarrow$$

$$p = \frac{3}{t}, \; g(t) = \frac{\cos 2t}{t^2}$$

An integrating factor satisfies

$$\mu' = \frac{3}{t}\mu \Rightarrow \mu = e^{\int \frac{3}{t} dt} = e^{3\ln t} = t^3 \Rightarrow$$

We can take

$$(\mu y)' = g(t)\mu(t) \Rightarrow$$

$$(t^3 y)' = \frac{\cos 2t}{t^2} \cdot t^3 = t\cos 2t \Rightarrow$$

$$t^3 y = \int t\cos 2t \, dt + C$$

$$\int t\cos 2t \, dt = \frac{1}{2} t \sin 2t - \frac{1}{2}\int \sin 2t \, dt = \frac{1}{2} t\sin 2t + \frac{1}{4}\cos 2t$$

integration by parts

$u = t$	$v = \frac{\sin 2t}{2}$
$u' = 1$	$v' = \cos 2t$

$$\Rightarrow \quad t^3 y = \frac{1}{2} t \sin 2t + \frac{1}{4}\cos 2t + C \Rightarrow \boxed{y(t) = \frac{\sin 2t}{2t^2} + \frac{1}{4}\frac{\cos 2t}{t^3} + \frac{C}{t^3}}$$

$\boxed{\lim_{t \to +\infty} y(t) = 0}$ because $\sin 2t$ and $\cos 2t$ are bounded and

$$\frac{1}{t^2} \xrightarrow[t \to +\infty]{} 0, \quad \frac{1}{t^3} \xrightarrow[t \to +\infty]{} 0$$

Problem 2

(a) $y' + 8y = te^{-7t}$, $y(0) = a$

$P = 8$, $g(t) = te^{-27}$

An integrating factor satisfies

$\mu' = 8\mu \Rightarrow$ We can take $\mu = e^{8t} \Rightarrow$

$(\mu y)' = te^{-7t}\mu \Leftrightarrow (e^{8t}y)' = te^{-7t}e^{8t} = te^{t}$

$e^{8t}y = \int te^{t}\,dt + C$

$\int te^{t}dt \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts}}}{=} te^{t} - \int e^{t}dt = te^{t} - e^{t} \Rightarrow$

$u = t \quad v = e^{t}$
$u' = 1 \quad v' = e^{t}$

$e^{8t}y = te^{t} - e^{t} + C \Rightarrow y(t) = te^{-7t} - e^{-7t} + Ce^{-8t}$

$y(0) = a \Rightarrow -1 + C = a \Rightarrow C = a + 1 \Rightarrow$

$$\boxed{y(t) = te^{-7t} - e^{-7t} + (a+1)e^{-8t}}$$

(b) $y(t) = te^{-7t}\left(1 - \frac{1}{t} + \frac{a+1}{t}e^{-t}\right)$

$te^{-7t} \underset{t \to -\infty}{\longrightarrow} -\infty$

$$1 - \frac{1}{t} + \frac{a+1}{t}e^{-t} \to \begin{cases} -\infty & \text{if } a+1 > 0 \\ +\infty & \text{if } a+1 < 0 \\ 1 & \text{if } a+1 = 0 \end{cases}$$

$$\Rightarrow y(t) \underset{t \to -\infty}{\longrightarrow} \begin{cases} +\infty & \text{if } a > -1 \quad \text{(Because we get } (-\infty)\times(-\infty) = +\infty \\ -\infty & \text{if } a = -1 \quad \text{(Because we get } (-\infty)\times 1 = -\infty \\ -\infty & \text{if } a < -1 \quad \text{(Because we get } (-\infty)\times(+\infty) = -\infty \end{cases}$$

Therefore $$\boxed{y(t) \underset{t \to -\infty}{\longrightarrow} \begin{cases} +\infty & \text{if } a > -1 \\ -\infty & \text{if } a \le -1 \end{cases}}$$

The value a_0 from which the transition occurs is $a_0 = -1$

(c) As already mentioned in the previous item for $a_0 = -1$

$$y(t) \underset{t\to-\infty}{\longrightarrow} -\infty$$

Problem 3 Let $Q(t)$ be the amount of salt at time t. Then $Q(0)=0$

1) For $0 \le t \le 30$

$$Q'(t) = \gamma r - \frac{Q(t)}{V} r =$$

$\gamma = 0.2$ lb/gal

$r = 30$ gal/min, $V = 600$ gal $\Rightarrow$

$$Q'(t) = 0.2\times 30 - \frac{Q(t)}{600}\cdot 30 = 6 - \frac{Q(t)}{20} \Rightarrow$$

$Q' + \frac{Q}{20} = 6$ $\quad p = \frac{1}{20}$ and the equation for the integrating factor is $\mu' = \frac{1}{20}\mu \Rightarrow \mu$ can be taken as $\mu = e^{\frac{t}{20}}$

$$\Rightarrow (\mu Q)' = 6\mu \Rightarrow \left(e^{\frac{t}{20}} Q\right)' = 6e^{\frac{t}{20}} \Rightarrow$$

$$e^{\frac{t}{20}} Q = \int 6e^{\frac{t}{20}}\,dt + C = \frac{6}{\frac{1}{20}} e^{\frac{t}{20}} + C = 120e^{\frac{t}{20}} + C$$

$\Rightarrow Q = 120 + Ce^{-\frac{t}{20}}$. Since $Q(0) = 120 + C = 0 \Rightarrow$

$C = -120 \Rightarrow Q(t) = 120 - 120e^{-\frac{t}{20}}$

Rem You could use the formula for the solution of the equation of the type $Q' = aQ + b$, where a and b are constant: $Q = -\frac{b}{a} + Ce^{at}$. Here $a = -\frac{1}{20}$, $b = 6$

$Q(30) = 120 - 120e^{-3/2}$ lb

2) For $t \geq 30$ $\quad \gamma = 0$, $r = 15\,\text{gal}/\text{min}$

$$\Downarrow$$

$$Q' = -\frac{Q(t)}{600} \cdot 15 = -\frac{Q(t)}{40} \Rightarrow Q(t) = Ce^{-\frac{t}{40}}$$

$$Q(30) = 120 - 120e^{-3/2} \text{ ex} \Rightarrow$$

$$120 - 120e^{-3/2} = Ce^{-3/4} \Rightarrow$$

$$C = 120e^{3/4} - 120e^{-3/4} = 120\left(e^{3/4} - e^{-3/4}\right)$$

$$\Downarrow$$

$$Q(t) = 120\left(e^{3/4} - e^{-3/4}\right)e^{-t/40}$$

We are interested in what happens after 20 min, i.e. for $t = 50$ min $\Rightarrow$ $Q(50) = 120\left(e^{3/4} - e^{-3/4}\right)e^{-5/4} =$

$$= \boxed{120\left(e^{-1/2} - e^{-2}\right)}$$