Homework Assignment #4 Solution Fall 2013 — MATH 308

Problem 1

$$(16-t^4)y' + (\ln(1-t))y = 13t^2 \iff$$

$$y' + \frac{\ln(1-t)}{16-t^4}\,y = \frac{13t^2}{16-t^4}$$

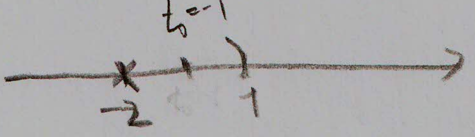
$$P(t) = \frac{\ln(1-t)}{16-t^4}, \quad g(t) = \frac{13}{16-t^4}$$

The domain of $P(t)$: $1-t>0$ and $16-t^4 \neq 0 \iff t<1$ and $t^4 \neq 16 \iff$

$t<1$ and $t \neq \pm 2 \Rightarrow$ the domain of $P(t)$

is $(-\infty,-2) \cup (-2,1)$

$g(t)$ is continuous for $t \neq \pm 2 \Rightarrow$



The maximal interval, containing $t_0=-1$ s.t. both $P(t)$ and $g(t)$ are continuous is

$(-2,1) \Rightarrow$ the solution is for sure defined for

$$\boxed{(-2,1)}$$

Problem 2

$$\underbrace{y\cos(xy) - e^{2y}}_{P(x,y)} + \underbrace{(x\cos(xy) - 2y - 2xe^{2y})}_{Q(x,y)}\,y' = 0$$

$P(x,y) = y\cos(xy) - e^{2y}$ $\quad P_y = \cos xy - xy\sin xy - 2e^{2y}$

$Q(x,y) = x\cos(xy) - 2y - 2xe^{2y}$ $\quad Q_x = \cos xy - xy\sin xy - 2e^{2y}$

$\iff P_y = Q_x \iff$ it is an exact equation.

Find the potential, i.e. a function

$$\begin{cases} \varphi_x = P \\ \varphi_y = Q \end{cases} \Leftrightarrow \begin{cases} \varphi_x = y\cos(xy) - e^{2y} & (1) \\ \varphi_y = x\cos(xy) - 2y - 2xe^{2y} & (2) \end{cases}$$

Integrating (1).

$$\varphi = \int \left(y\cos(xy) - e^{2y}\right) dx + h(y) =$$

$$= \frac{\cancel{y}\sin(xy)}{\cancel{y}} - xe^{2y} + h(y)$$

Substituting this into (2).

$$\cancel{x\cos(xy)} - \cancel{2xe^{2y}} + h'(y) = \cancel{x\cos(xy)} - 2y - \cancel{2xe^{2y}} \Rightarrow$$

$$h'(y) = -2y \Rightarrow h(y) = -y^2 + C$$

We can take $C=0$

$$\varphi = \sin(xy) - xe^{2y} - y^2 \Rightarrow$$

The general solution is given implicitly by

$$\boxed{\sin(xy) - xe^{2y} - y^2 = C}$$

3. $$6xy\,dx + (4y + 9x^2)\,dy = 0$$

$P = 6xy \qquad P_y = 6x$

$Q = 4y + 9x^2 \qquad Q_x = 18x \qquad P_y \neq Q_x$ – not exact

$$\frac{P_y - Q_x}{Q} = \frac{6x - 18x}{4y + 9x^2}$$ depend both on x and y $\Rightarrow$ the integrating factor μ cannot be found as the function of μ only

$$\frac{Q_x - P_y}{P} = \frac{18x-6x}{6xy} = \frac{12xy}{6xy} = \frac{2}{y} \text{ is}$$

the function of y only $\Rightarrow$

We need $(\mu(y) P)_y = (\mu(y) Q)_x \Rightarrow$

$$\frac{d}{dy}\mu \, P + \mu(y) P_y = \mu(y) Q_x \Rightarrow$$

$$\frac{d}{dy}\mu = \frac{Q_x - P_y}{P}\mu = \frac{2}{y}\mu \Rightarrow$$

μ can be taken as $\mu = e^{\int \frac{2}{y} dy} = e^{\ln\left|\frac{2}{y}\right|} = y^2$

Multiply our equation by y^2:

$$\underbrace{6xy^3}_{P} dx + \underbrace{(4y^3 + 9x^2y^2)}_{Q} dy = 0$$

[Check of exactness (not necessary to do)

$P_y = 18xy^2$

$Q_x = 18xy^2$ $\Rightarrow P_y = Q_x$]

$$\begin{cases} \varphi_x = 6xy^3 \Rightarrow \varphi = \int 6xy^3 dx + h(y) = 3x^2y^3 + h(y). \\ \varphi_y = 4y^3 + 9x^2y^2 \end{cases}$$

Substituting to the 2nd eq: $\cancel{9x^2y^2} + h'(y) = 4y^3 + \cancel{9x^2y^2}$

$\Rightarrow h'(y) = 4y^3 \Rightarrow h(y) = \int 4y^3 dy + C = y^4 + C$

Take $C=0$. $\Rightarrow \varphi = 3x^2y^3 + y^4 \Rightarrow$ the general solution is given implicitly by $\boxed{3x^2y^3 + y^4 = C}$