Homework Assignment #5 MATH 308-505 FALL 2013
Solutions

1 (a) $y_1(t) = \frac{1}{t}\cos(2\ln t) \Rightarrow y_1'(t) = -\frac{1}{t^2}\cos(2\ln t) -$

$- \frac{1}{t}\sin(2\ln t)\cdot\frac{2}{t} = -\frac{1}{t^2}\cos(2\ln t) - \frac{2}{t^2}\sin(2\ln t)$

$y_2(t) = \frac{1}{t}\sin(2\ln t) \Rightarrow y_2'(t) = -\frac{1}{t^2}\sin(2\ln t) + \frac{1}{t}\cos(2\ln t)\frac{2}{t} =$

$= -\frac{1}{t^2}\sin(2\ln t) + \frac{2}{t^2}\cos(2\ln t) \Rightarrow$

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \frac{1}{t}\cos(2\ln t) & \frac{1}{t}\sin(2\ln t) \\ -\frac{1}{t^2}\cos(2\ln t) - \frac{2}{t^2}\sin(2\ln t) & -\frac{1}{t^2}\sin(2\ln t) + \frac{2}{t^2}\cos(2\ln t) \end{vmatrix}$$

$= -\frac{1}{t^3}\cos(2\ln t)\sin(2\ln t) + \frac{2}{t^3}\cos^2(2\ln t) + \frac{1}{t^3}\sin(2\ln t)\cos(2\ln t) +$

$+ \frac{2}{t^3}\sin^2(2\ln t) = \frac{2}{t^3}\underbrace{(\cos^2(2\ln t) + \sin^2(2\ln t))}_{=1} = \boxed{\frac{2}{t^3}} \neq 0$

1(b) i) Substitute $y_1(t)$ into equation (1)

$y_1(t) = \frac{1}{t}\cos(2\ln t)$

$y_1'(t) = -\frac{1}{t^2}\cos(2\ln t) - \frac{2}{t^2}\sin(2\ln t)$

$y_1''(t) = \frac{2}{t^3}\cos(2\ln t) + \frac{2}{t^3}\sin(2\ln t) + \frac{4}{t^3}\sin(2\ln t) - \frac{4}{t^3}\cos(2\ln t)$

$= -\frac{2}{t^3}\cos(2\ln t) + \frac{6}{t^3}\sin(2\ln t)$

$\Rightarrow t^2y_1'' + 3ty_1' + 5y_1 = -\frac{2}{t}\cos(2\ln t) + \frac{6}{t}\sin(2\ln t) - \frac{3}{t}\cos(2\ln t) -$

$- \frac{6}{t}\sin(2\ln t) + \frac{5}{t}\cos(2\ln t) = 0 \Rightarrow y_1(t)$ is a solution

ii) Substitute $y_2(t)$ into equation (1)

$$y_2(t) = \frac{1}{t}\sin(2\ln t)$$

$$y_2'(t) = -\frac{1}{t^2}\sin(2\ln t) + \frac{2}{t^2}\cos(2\ln t)$$

$$y_2''(t) = \frac{2}{t^3}\sin(2\ln t) - \frac{2}{t^3}\cos(2\ln t) - \frac{4}{t^3}\cos(2\ln t) - \frac{4}{t^3}\sin(2\ln t) =$$

$$= -\frac{2}{t^3}\sin(2\ln t) - \frac{6}{t^3}\cos(2\ln t)$$

$$t^2 y_2''(t) + 3t\, y_2'(t) + 5y_2(t) = -\frac{2}{t}\sin(2\ln t) - \frac{6}{t}\cos(2\ln t) - \frac{3}{t}\sin(2\ln t) + \frac{6}{t}\cos(2\ln t) + \frac{5}{t}\sin(2\ln t) = 0$$

$\Rightarrow y_2(t)$ is a solution

(c) By (a) & (b) $y_1(t)$ and $y_2(t)$ are solutions of (1) with <u>nonzero Wronskian</u> $\Rightarrow$ they constitute the fundamental set of solutions $\Rightarrow$ the general solution of (1) is

$$\boxed{y(t) = C_1 t^{-1}\cos(2\ln t) + C_2 t^{-1}\sin(2\ln t)}$$

(d) Determine C_1 and C_2 so that $y(t)$ satisfies the given initial conditions

$$2 = y(1) = C_1 \cdot \frac{1}{1}\underbrace{\cos(\underbrace{2\ln 1}_{0})}_{1} + C_2\,\frac{1}{1}\underbrace{\sin(\underbrace{2\ln 1}_{0})}_{0} = C_1 \Rightarrow$$

$C_1 = 2$

$$y'(t) = 2 \cdot \underbrace{\left(-\frac{1}{t^2}\cos(2\ln t) - \frac{2}{t^2}\sin(2\ln t)\right)}_{y_1'(t)} +$$

$$+ C_2 \underbrace{\left(-\frac{1}{t^2}\sin(2\ln t) + \frac{2}{t^2}\cos(2\ln t)\right)}_{y_2'(t)} \Rightarrow$$

$y_1'(1) = -1$, $y_2'(1) = 2$

$-4 = y'(1) = -2 + 2C_2 \Rightarrow 2C_2 - 2 = -4 \Rightarrow 2C_2 = -2 \Rightarrow$

$C_2 = -1 \Rightarrow$

$$\boxed{y(t) = 2t^{-1}\cos(2\ln t) - t^{-1}\sin(2\ln t)}$$

Problem 2 $\quad y'' + 3y' - 10y = 0$

The characteristic equation is

$r^2 + 3r - 10 = 0 \Rightarrow$

$D = b^2 - 4ac = 9 - 4\cdot 1\cdot(-10) = 9 + 40 = 49 > 0$

$r_1 = \dfrac{-3+7}{2} = 2$

$r_2 = \dfrac{-3-7}{2} = -5$

$\Rightarrow$ the general solution is $\boxed{y(t) = C_1 e^{2t} + C_2 e^{-5t}}$

(b) $y(0) = \alpha \Rightarrow \begin{cases} C_1 + C_2 = \alpha \\ 2C_1 - 5C_2 = 1 \end{cases}$ $\overset{Eq2 - 2Eq1}{\Longrightarrow}$ $-7C_2 = 1 - 2\alpha \Rightarrow$

$y'(0) = 1 \Rightarrow$ $\qquad C_2 = \dfrac{2\alpha - 1}{7}$

$5\,Eq\,1 + Eq\,2 \Rightarrow 7C_1 = 5\alpha + 1 \Rightarrow C_1 = \frac{5\alpha+1}{7}$

$\Downarrow$

$$\boxed{y(t) = \frac{5\alpha+1}{7} e^{2t} + \frac{2\alpha-1}{7} e^{-5t}}$$

(c) $y(t) \underset{t\to+\infty}{\longrightarrow} 0 \Leftrightarrow$ the coefficient of e^{2t} is equal to $0 \Leftrightarrow$

$$\frac{5\alpha+1}{7} = 0 \Leftrightarrow \boxed{\alpha = -\frac{1}{5}}$$

(d) $y(t) \underset{t\to-\infty}{\longrightarrow} 0 \Leftrightarrow$ the coefficient of e^{-5t} is equal to $0 \Leftrightarrow \frac{2\alpha-1}{7} = 0 \Leftrightarrow \boxed{\alpha = \frac{1}{2}}$

<u>Problem 3</u> Our equation can be written as

$$y'' + \frac{3}{t} y' + \frac{\sin(e^t)}{t} y = 0, \quad t > 0$$

By Abel's theorem $W(t) := W(y_1, y_2)(t)$ satisfies

$$W'(t) = -\frac{3}{t} W(t) \Rightarrow W(t) = C e^{\int -\frac{3}{t}\,dt} = C e^{-3\ln t} = Ct^{-3}$$

Find C: $W(3) = 4 \Rightarrow 4 = C \cdot 3^{-3} = \frac{C}{27} \Rightarrow C = 27 \times 4 = 108$

$\Rightarrow W(t) = 108\,t^{-3} \Rightarrow W(5) = 108 \cdot 5^{-3} = \boxed{\frac{108}{125}}$