MATH308 Homework 5 Solutions

1) (a) $y''-3y'-10y=4e^{5t}$

i) First, find the fundamental set of solutions of the corresponding homogeneous equation

$$y''-3y'-10y=0$$

Characteristic equation: $r^2-3r-10=0$

$$D=9-4\cdot(-10)=9+40=49$$
$$r_1=\frac{3+7}{2}=5$$
$$r_2=\frac{3-7}{2}=-2$$
$$\Downarrow$$

$\{e^{5t}, e^{-2t}\}$ is the fundamental set of solutions of the homogeneous equations

ii) According to the method of variation of parameters we look for a solution of the nonhomogeneous equation in the form

$$y(t)=u_1(t)e^{5t}+u_2(t)e^{-2t}$$

such that $u_1'(t)$ and $u_2'(t)$ satisfy the following system of linear equations:

$$\begin{cases} e^{5t}u_1'+e^{-2t}u_2'=0 \\ 5e^{5t}u_1'-2e^{-2t}u_2'=4e^{5t} \end{cases} \xRightarrow{\text{By Cramer's rule}}$$

$$u_1'=\frac{\begin{vmatrix} 0 & e^{-2t} \\ 4e^{5t} & -2e^{-2t} \end{vmatrix}}{\begin{vmatrix} e^{5t} & e^{-2t} \\ 5e^{5t} & -2e^{-2t} \end{vmatrix}}=\frac{-4e^{3t}}{(-2-5)e^{3t}}=\frac{4}{7}\Rightarrow u_1=\frac{4}{7}t+\tilde{C}_1$$

$$u_2'=\frac{\begin{vmatrix} e^{5t} & 0 \\ 5e^{5t} & 4e^{5t} \end{vmatrix}}{(-2-5)e^{3t}}=\frac{4e^{10t}}{-7e^{3t}}=-\frac{4}{7}e^{7t}\Rightarrow u_2=-\frac{4}{49}e^{7t}+C_2$$

$$\Rightarrow y(t) = u_1(t)e^{5t} + u_2(t)e^{-2t} = \left(\frac{4}{7}t + \tilde{C}_1\right)e^{5t} + \left(-\frac{4}{49}e^{7t} + C_2\right)e^{-2t} =$$

$$= \frac{4}{7}te^{5t} + \underbrace{\left(\tilde{C}_1 - \frac{4}{49}\right)}_{C_1}e^{5t} + C_2e^{-2t} = \boxed{\frac{4}{7}te^{5t} + C_1e^{5t} + C_2e^{-2t}}$$

1(B) $y'' - 2y' + y = \frac{e^t}{t}$, $t > 0$

(i) First, find the fundamental set of solutions of the corresponding homogeneous equation:

$$y'' - 2y' + y = 0$$

Characteristic equation: $r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Rightarrow r = 1 \rightarrow$ repeated root $\Rightarrow$

$\{e^t, te^t\}$ is the fundamental set of solutions of the corresponding homogeneous equation

(ii) According to the method of variation of parameters we look for a solution of the nonhomogeneous equation in the form

$$y(t) = u_1(t)e^t + u_2(t)te^t$$

such that $u_1'(t)$ and $u_2'(t)$ satisfy the following system of linear equations:

$$\begin{cases} e^t u_1'(t) + te^t u_2'(t) = 0 \\ e^t u_1'(t) + (e^t + te^t)u_2'(t) = \frac{e^t}{t} \end{cases} \xRightarrow{\text{By Cramer's rule}}$$

$$u_1'(t) = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t} & e^t + te^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{-te^t \cdot \frac{e^t}{t}}{e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}}} = -\frac{e^{2t}}{e^{2t}} = -1 \Rightarrow$$

$$u_1(t) = (-t + C_1)$$

$$u_2'(t) = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{t} \end{vmatrix}}{e^{2t}} = \frac{e^{2t}}{t e^{2t}} = \frac{1}{t} \Rightarrow u_2(t) = \ln t + C_2$$

$$\Downarrow$$

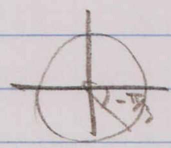
$$y(t) = u_1(t)e^t + u_2(t)te^t = (-t + C_1)e^t + (\ln t + C_2)te^t =$$

$$= \boxed{(-t + t\ln t + C_1 + C_2 t)e^t} \quad \text{or} \quad \boxed{(t(\ln t - 1) + C_1 + C_2 t)e^t}$$

2a $\cos 4t - \sqrt{3}\sin 4t \Rightarrow \boxed{\omega_0 = 4}$ and $C_1 = 1,\ C_2 = -\sqrt{3}$

$$R = \sqrt{C_1^2 + C_2^2} = \sqrt{1+3} = \sqrt{4} = 2 \Rightarrow \boxed{R=2}$$

$$\begin{cases} \cos\delta = \frac{C_1}{R} = \frac{1}{2} \\ \sin\delta = \frac{C_2}{R} = -\frac{\sqrt{3}}{2} \end{cases} \Rightarrow \text{$\delta$ is in forth quadrant} \Rightarrow \boxed{\delta = -\frac{\pi}{3}}$$




2b $\cos 5t - \cos 2t = -2\sin\frac{5t-2t}{2}\sin\frac{5t+2t}{2} = \boxed{-2\sin\frac{3t}{2}\sin\frac{7t}{2}}$

3a First, find the spring coefficient $k$ and the mass $m$

$$\overset{mg}{W} = kL \Rightarrow k = \frac{W}{L} = \frac{8\text{ lb}}{\frac{1}{2}\text{ft}} = 16\ \frac{\text{lb}}{\text{ft}}$$

$L = 6\text{in} = \frac{1}{2}\text{ft}$

$$m = \frac{W}{g} = \frac{8}{32}\ \frac{\text{lb}\cdot s^2}{\text{ft}} = \frac{1}{4}\ \frac{\text{lb}\cdot s^2}{\text{ft}}$$

The equation of the sping is $mu'' + ku = 0$, or

$$\frac{1}{4}u'' + 16u = 0 \Leftrightarrow u'' + 64u = 0$$

Initial conditions: $u(0) = -3\text{in} = -\frac{1}{4}\text{ft},\ u'(0) = 0$

$$\boxed{\begin{array}{l}\text{The frequency} = \sqrt{\frac{k}{m}} = \sqrt{64} = 8 \\ \text{The period} = \frac{2\pi}{8} = \frac{\pi}{4}\end{array}}$$

$u(t) = C_1 \cos 8t + C_2 \sin 8t$

Find $C_1$ and $C_2$ from the initial condition

$u(0) = C_1 = -\frac{1}{4}$

$u'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t \Rightarrow u'(0) = 8C_2 = 0 \Rightarrow \boxed{C_2 = 0}$

$\boxed{u(t) = -\frac{1}{4}\cos 8t} \Rightarrow$ The amplitude is $\boxed{\frac{1}{4}\text{ ft}}$

(b) $\gamma_{crit} = 2\sqrt{km} = 2\sqrt{\underbrace{16 \cdot \frac{1}{4}}_{4}}\ \frac{lb \cdot s}{ft} = \boxed{4}$

4. (a) First, find the spring coefficient k and the mass m

$W = kL \Rightarrow k = \frac{W}{L} = \frac{2\,lb}{1/4\,ft} = 8\,\frac{lb}{ft}$

$L = 3\,in = 1/4\,ft$

$m = \frac{W}{g} = \frac{2}{32}\,\frac{lb \cdot s^2}{ft} = \frac{1}{16}\,\frac{lb \cdot s^2}{ft}$

The equation of the spring is $mu'' + \gamma u' + ku = F_{ext}$

$\frac{1}{16}u'' + 2u' + 8u = 2\cos 3t$

The characteristic equation is

$\frac{1}{16}r^2 + 2r + 8 = 0$

$D = 4 - 2 = 2$

$r_{1,2} = \frac{-2 \pm \sqrt{2}}{\frac{1}{8}} = -16 \pm 8\sqrt{2} \rightarrow$ here we got even real roots so we get Over damping

In any case the steady state solution has the form

$u(t) = R\cos(3t - \delta)$, where

*) $R = \dfrac{2}{\Delta}$ with $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ (see page 208, it also was given in class)

Here $\omega = 3$, $\omega_0^2 = \dfrac{k}{m} = \dfrac{8}{\frac{1}{16}} = 128$, $m = \dfrac{1}{16}$, $\gamma = 2 \Rightarrow$

$$\Delta = \sqrt{\frac{1}{16^2}(128-9)^2 + 4\cdot 9} = \boxed{\sqrt{\left(\frac{119}{16}\right)^2 + 36} = \frac{\sqrt{23377}}{16}} \approx 9.55$$

$$\Downarrow$$

$$\boxed{R = \frac{32}{\sqrt{23,377}}} \approx 0.21 \text{ ft}$$

**) $\cos\delta = \dfrac{\frac{1}{16}(128-9)}{\frac{\sqrt{23.377}}{16}} = \boxed{\dfrac{119}{\sqrt{23.377}}} \simeq 0.77$

$\Rightarrow \delta$ is in the first quadrant

$\sin\delta = \dfrac{2\cdot 3}{\frac{\sqrt{23,377}}{16}} = \dfrac{96}{\sqrt{23,377}}$

and $\boxed{\delta = \arccos\dfrac{119}{\sqrt{23,377}}} \simeq 38.83^\circ$

$$\boxed{u(t) = \frac{32}{\sqrt{23,377}}\cos\left(3t - \arccos\frac{119}{\sqrt{23,377}}\right)}$$

(b) $$\omega_{max}^2 = \begin{cases} \omega_0^2 - \dfrac{\gamma^2}{2m^2} & \text{if } \omega_0^2 - \dfrac{\gamma^2}{2m^2} > 0 \\ 0 & \text{if } \omega_0^2 - \dfrac{\gamma^2}{2m^2} \leq 0 \end{cases}$$ (see page 209, it was discussed in class)

In our case $\omega_0^2 - \dfrac{\gamma^2}{2m^2} = 128 - \dfrac{4}{2\cdot\frac{1}{16^2}} = 128 - 512 < 0 \Rightarrow$

In fact $\boxed{\omega_{max} = 0}$ So here the damping factor is too big to have a nontrivial answer: the external force giving the maximal amplitude is the constant force.

**Remark** Actually the condition $\omega > 0$ should be removed from the formulation of this item.

(5) (a) $u'' + 100u = 3\cos 9t$, $u(0)=0$, $u'(0)=0$

$\omega_0 = 10$, $\omega = 9$, $\frac{F_0}{m} = 3$

$$u(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t) = \frac{3}{100-81}(\cos 9t - \cos 10t) =$$

(see page 213, formula (20), it was given in class)

$$= \underbrace{\frac{6}{19}\sin\frac{t}{2}}_{\text{amplitude modulation or a beat}} \sin\frac{19t}{2}$$

5 (b) and (c) see the files posted next to this file: 3 files with the program, result for 5a and figure for 5b.