Homework #6 Solutions, FALL 2013 - MATH 308

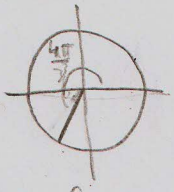
Problem 1 (a) $(-2+3i)(3-i)i = (-6+9i+2i+3)i = (-3+11i)i = -11-3i$

(b) $\frac{5+2i}{3+4i} = \frac{(5+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{15+6i-20i+8}{25} = \boxed{\frac{23}{25} - \frac{14}{25}i}$

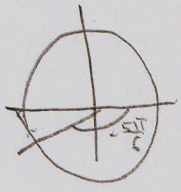
$(3+4i)(3-4i) = 3^2+4^2$

(in general $z \cdot \bar{z} = |z|^2$)

(c) $e^{\frac{4\pi i}{3}} = \underbrace{\cos\frac{4\pi}{3}}_{-\frac{1}{2}} + i\underbrace{\sin\frac{4\pi}{3}}_{-\frac{\sqrt{3}}{2}} = \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$



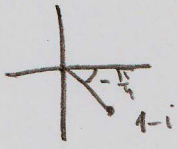
(d) $e^{\left(2013 - \frac{5\pi}{6}i\right)} = e^{2013}\left(\underbrace{\cos\left(-\frac{5\pi}{6}\right)}_{-\frac{\sqrt{3}}{2}} + i\underbrace{\sin\left(-\frac{5\pi}{6}\right)}_{-\frac{1}{2}}\right) = \boxed{-\frac{\sqrt{3}}{2}e^{2013} - i\frac{e^{2013}}{2}}$



(e) $(1-i)^8$

Use polar representation $|1-i| = \sqrt{2} \Rightarrow 1-i = \sqrt{2}e^{-\frac{\pi}{4}i} \Rightarrow$

$\arg(1-i) = -\frac{\pi}{4}$



$(1-i)^8 = \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^8 = (\sqrt{2})^8\left(e^{-i\frac{\pi}{4}}\right)^8 =$

$= 16 \cdot \underbrace{e^{-2\pi i}}_{1} = \boxed{16}$

Problem 2 $4y'' + 16y' + 25y = 0$

(a) General solution

$D = 16^2 - 4 \cdot 4 \cdot 25 = 256 - 400 = -144 < 0$

$r_{1,2} = \frac{-16 \pm i\sqrt{144}}{8} = -2 \pm i\frac{12}{8} = -2 \pm i\frac{3}{2} \Rightarrow$

$\lambda = -2$, $\mu = \frac{3}{2}$

$$\boxed{y(t) = C_1 e^{-2t} \cos\frac{3}{2}t + C_2 e^{-2t} \sin\frac{3}{2}t}$$

(b) $y\left(\frac{2\pi}{3}\right) = -1 \Rightarrow C_1 e^{-\frac{4\pi}{3}} \underbrace{\cos\left(\overbrace{\frac{3}{2}\cdot\frac{2}{3}\pi}^{\pi}\right)}_{-1} + C_2 e^{-\frac{4\pi}{3}} \underbrace{\sin\left(\overbrace{\frac{3}{2}\cdot\frac{2}{3}\pi}^{\pi}\right)}_{0} =$

$= -C_1 e^{-\frac{4\pi}{3}} = -1 \Rightarrow \boxed{C_1 = e^{\frac{4\pi}{3}}}$

$y'(t) = -2C_1 e^{-2t} \cos\frac{3}{2}t - \frac{3}{2} C_1 e^{-2t} \sin\frac{3}{2}t - 2C_2 e^{-2t} \sin\frac{3}{2}t +$

$+ \frac{3}{2} C_2 e^{-2t} \cos\frac{3}{2}t$

$y'\left(\frac{2\pi}{3}\right) = -2C_1 e^{-\frac{4\pi}{3}} \underbrace{\cos\pi}_{-1} - \frac{3}{2} C_1 e^{-\frac{4\pi}{3}} \underbrace{\sin\pi}_{0} - 2C_2 e^{-\frac{4\pi}{3}} \underbrace{\sin\pi}_{0} +$

$+ \frac{3}{2} C_2 e^{-\frac{4\pi}{3}} \underbrace{\cos\pi}_{-1} = 2\underbrace{C_1}_{e^{\frac{4\pi}{3}}} e^{-\frac{4\pi}{3}} - \frac{3}{2} C_2 e^{-\frac{4\pi}{3}} = -2 \Rightarrow$

$2 - \frac{3}{2} C_2 e^{-\frac{4\pi}{3}} = -2 \Rightarrow \frac{3}{2} C_2 e^{-\frac{4\pi}{3}} = 4 \Rightarrow \boxed{C_2 = \frac{8}{3} e^{\frac{4\pi}{3}}}$

$\Downarrow$

$$\boxed{y(t) = e^{\frac{4\pi}{3} - 2t} \cos\frac{3}{2}t + \frac{8}{3} e^{\frac{4\pi}{3} - 2t} \sin\frac{3}{2}t} =$$

$= e^{-2t}\left(e^{\frac{4\pi}{3}} \cos\frac{3}{2}t + \frac{8}{3} e^{\frac{4\pi}{3}} \sin\frac{3}{2}t\right) \xrightarrow[t \to +\infty]{} \boxed{0}$