Homework #6 MATH 308 solutions

Spn

Problem 1 (a)

$$AB = \begin{pmatrix} 3 & -8 & 4 \\ -1 & 5 & -6 \\ 7 & 5 & -6 \end{pmatrix} \begin{pmatrix} 3 & 3 & -2 \\ 4 & -5 & 1 \\ 4 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 9-32+16 & 9+40-8 & -6-8+12 \\ -3+20-24 & -3-25+12 & 2+5-18 \\ 21+20-24 & 21-25+12 & -14+5-18 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 41 & -2 \\ -7 & -16 & -11 \\ 17 & 8 & -27 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 3 & -2 \\ 4 & -5 & 1 \\ 4 & -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -8 & 4 \\ -1 & 5 & -6 \\ 7 & 5 & -6 \end{pmatrix} = \begin{pmatrix} 9-3-14 & -24+15-10 & 12-18+12 \\ 12+5+7 & -32-25+5 & 16+30-6 \\ 12+2+21 & -32-10+15 & 16+12-18 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & -19 & 6 \\ 24 & -52 & 40 \\ 35 & -27 & 10 \end{pmatrix}$$

$$AB - 3BA = \begin{pmatrix} -7+24 & 41+57 & -2-18 \\ -7-72 & -16+156 & -11-120 \\ 17-105 & 8+81 & -27-30 \end{pmatrix} =$$

$$= \begin{pmatrix} 17 & 98 & -20 \\ -79 & 140 & -131 \\ -88 & 89 & -57 \end{pmatrix}$$

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(l)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & & a_{3n} \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{pmatrix}$$

$\therefore \det A = a_{11} a_{22} \cdots a_{nn}$

Proof Way 1 Decompose w.r.t. the first column or the first row

$$\det A = a_{11} \det \begin{pmatrix} a_{22} & a_{21} & \cdots & a_{2n} \\ & a_{33} & \cdots & a_{3n} \\ 0 & & \ddots & a_{nn} \end{pmatrix} \overset{=}{\text{similarly to the previous}}$$

$$= a_{11} a_{22} \det \begin{pmatrix} a_{33} & \cdots & a_{3n} \\ 0 & \ddots & a_{nn} \end{pmatrix} = \cdots =$$

$$= a_{11} a_{22} \cdots a_{nn}$$

Way 2 Using the formula for the determinant

$$\det A = \sum_{\substack{\text{all} \\ \text{permutations} \\ \tau \text{ of } \{1,\dots,n\}}} (-1)^{\operatorname{sgn} \tau} a_{1\tau(1)} \cdots a_{n\tau(n)} =$$

$$= \underbrace{a_{11} \cdots a_{nn}}_{\text{The only nonzero term in this sum}}$$

because other terms must contain some entries below the diagonal.

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ii. Characteristic equation for A is

$$\det(A - \lambda I) = 0$$

$$\det(A-\lambda I) = \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & & a_{1n} \\ 0 & a_{22}-\lambda & & & a_{2n} \\ & & a_{33}-\lambda & & \\ & 0 & & \ddots & \\ & & 0 & & a_{nn}-\lambda \end{pmatrix} =$$

$$\overset{\text{by i)}}{=} (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0 \Rightarrow$$

by i) applied to the upper triangular matrix $A - \lambda I$

The eigenvalues of A are $a_{11}, a_{22}, \ldots, a_{nn}$, i.e. the diagonal elements.

iii) $$Av^1 = \begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ & a_{22} & & a_{2n} \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$$

$= a_{11} v^1 \Rightarrow v$ is an eigenvector with eigenvalue a_{11}

iv) $$Av^i = \begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ & a_{22} & & a_{2n} \\ 0 & & \ddots & \cdots \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow \text{on } i\text{th entry} \quad =$$

$$= \underbrace{\begin{pmatrix} a_{1i} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{the ith column of i}}$$ and it is of the form $\lambda v^i = \begin{pmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{pmatrix}$

if and only if $\boxed{a_{1i} = a_{2i} = \dots = a_{i-1\,i} = 0}$

Problem 2

$$\begin{cases} x_1' = -6x_1 - 2x_2 - 5x_3 \\ x_2' = -3x_1 + x_2 + x_3 \\ x_3' = 12x_1 + 2x_2 + 11x_3 \end{cases}$$

(a) 1) Plug $x^1(t) = (e^{3t}, -2e^{3t}, -e^{3t})^T$ into the system

.H.S: $\frac{d}{dt} x^1(t) = \begin{pmatrix} 3e^{3t} \\ -6e^{3t} \\ -3e^{3t} \end{pmatrix}$

R.H.S: $\begin{pmatrix} -6e^{3t} + 4e^{3t} + 5e^{3t} \\ -3e^{3t} - 2e^{3t} - e^{3t} \\ 12e^{3t} - 4e^{3t} - 11e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -6e^{3t} \\ -3e^{3t} \end{pmatrix}$

$\Rightarrow$ L.H.S = R.H.S i.e $x^1(t)$ is a solution

2) Plug $x^2(t) = (-e^{6t}, e^{6t}, 2e^{6t})^T$ into the system

L.H.S: $\frac{d}{dt} x^2(t) = \begin{pmatrix} -6e^{6t} \\ 6e^{6t} \\ 12e^{6t} \end{pmatrix}$

R.H.S: $\begin{pmatrix} +6e^{6t} - 2e^{6t} - 10e^{6t} \\ 3e^{6t} + e^{6t} + 2e^{6t} \\ -12e^{6t} + 2e^{6t} + 22e^{6t} \end{pmatrix} = \begin{pmatrix} -6e^{6t} \\ 6e^{6t} \\ 12e^{6t} \end{pmatrix}$ => L.H.S.=R.H.S i.e $x^2(t)$ is a solution

3) $x^3(t) = \begin{pmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}$

L.H.S $\frac{d}{dt}x^3(t) = \begin{pmatrix} 3e^{-3t} \\ 3e^{-3t} \\ -3e^{-3t} \end{pmatrix}$ => L.H.S=R.H.S i.e $x^3(t)$ is a solution

R.H.S: $\begin{pmatrix} 6e^{-3t} + 2e^{-3t} - 5e^{-3t} \\ 3e^{-3t} - e^{-3t} + e^{-3t} \\ -12e^{-3t} - 2e^{-3t} + 11e^{-3t} \end{pmatrix} = \begin{pmatrix} 3e^{-3t} \\ 3e^{-3t} \\ -3e^{-3t} \end{pmatrix}$

(b) To check whether $x^1(t)$, $x^2(t)$ & $x^3(t)$ constitute a fundamental set of solutions we should check whether its Wronskian is not zero and it is sufficient to check at one point $t_0 = t_0$

It is more simple to take $t=0$;

$$\det\begin{pmatrix} x^1(0) & x^2(0) & x^3(0) \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} =$$

$$= 1\,(1+2) + 1\,(-2-1) - 1\,(-4+1) = 3-3+3 = 3 \neq 0$$

=> $x^1(t), x^2(t), x^3(t)$ is a fundamental set of solutions =>

The general solution is $x(t) = c_1 x^1(t) + c_2 x^2(t) + c_3 x^3(t) =$

$$= C_1 \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + C_2 \begin{pmatrix} -e^{6t} \\ e^{6t} \\ 2e^{6t} \end{pmatrix} + C_3 \begin{pmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}$$

(c) $x^1(t) = e^{3t} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \Rightarrow \lambda_1 = 3$ is an eigenvalue with an eigenvector $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$x^2(t) = e^{6t} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \lambda_2 = 6$ is an eigenvalue with an eigenvector $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$x^3(t) = e^{-3t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \lambda_3 = -3$ is an eigenvalue with an eigenvector $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Problem 3 $\begin{cases} x_1' = -16x_1 + 9x_2 \\ x_2' = -30x_1 + 17x_2 \end{cases}$

(a) $A = \begin{pmatrix} -16 & 9 \\ -30 & 17 \end{pmatrix}$ $\quad \operatorname{tr} A = -16 + 17 = 1$

$\det A = \underbrace{-16 \cdot 17}_{-272} + \underbrace{9 \cdot 30}_{270} = -2$

$\lambda^2 - \lambda - 2 = 0 \Rightarrow D = 1 + 2 \cdot 4 = 9$ $\quad \left.\begin{array}{l} \lambda_1 = \frac{1+3}{2} = 2 \\ \lambda_2 = \frac{1-3}{2} = -1 \end{array}\right\}$ distinct real eigenvalues

- Find an eigenvector for $\lambda = 2$

$(A - 2I)v = \begin{pmatrix} -18 & 9 \\ -30 & 15 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -2v_1 + v_2 = 0$ Take $v_1 = 1 \Rightarrow v_2 = 2$

[illegible] $\Rightarrow \quad v^1 = \begin{pmatrix}1\\2\end{pmatrix} \Rightarrow e^{2t}\begin{pmatrix}1\\2\end{pmatrix}$ is a solution

- Find an eigenvector for $\lambda = -1$

$$(A-(-I))v = (A+I)v = \begin{pmatrix}-15 & 9\\ -30 & 18\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Leftrightarrow \quad -5v_1+3v_2=0 \quad \Rightarrow \text{ let } v_2=5 \Rightarrow v_1=3$$

$$v^2 = \begin{pmatrix}3\\5\end{pmatrix} \quad \Rightarrow e^{-t}\begin{pmatrix}3\\5\end{pmatrix} \text{ is a solution}$$

$\Downarrow$

The general solution is $\boxed{C_1 e^{2t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}3\\5\end{pmatrix}}$

(b) $\quad x_1(0) = -7, \quad x_2(0) = -11 \Rightarrow$

$$C_1\begin{pmatrix}1\\2\end{pmatrix} + C_2\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-7\\-11\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1 & 3 & -7\\ 2 & 5 & -11\end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c}1 & 3 & -7\\ 0 & -1 & 3\end{array}\right) \Rightarrow \begin{array}{l} C_1+3C_2=-7 \\ -C_2 = 3 \Rightarrow C_2=-3\end{array}$$

$$C_1 - 9 = -7 \Rightarrow C_1 = 2 \quad \Rightarrow x(t) = 2e^{2t}\begin{pmatrix}1\\2\end{pmatrix} - 3e^{-t}\begin{pmatrix}3\\5\end{pmatrix}$$

(c) $x(t) = C_1 e^{2t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}3\\5\end{pmatrix} \xrightarrow[t\to\infty]{} 0 \quad \Leftrightarrow C_1 = 0$

$$\Rightarrow \quad \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = X(0) = C_2\begin{pmatrix}3\\5\end{pmatrix} \Leftrightarrow \boxed{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \text{ is collinear to } \begin{pmatrix}3\\5\end{pmatrix}}$$

or $\boxed{5\alpha_1 - 3\alpha_2 = 0}$

1d) $x(t) = C_1 e^{2t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}3\\5\end{pmatrix} \xrightarrow[t\to-\infty]{} 0 \iff C_1 = 0$

$$\Rightarrow \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = x_{+}(0) = C_1\begin{pmatrix}1\\2\end{pmatrix} \iff \boxed{\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} \text{ is collinear to } \begin{pmatrix}1\\2\end{pmatrix}}$$

or $\boxed{2\beta_1 - \beta_2 = 0}$

Problem 4 $\begin{cases} x_1' = 7x_1 + 11x_2 + 3x_3 \\ x_2' = -4x_1 - 8x_2 - 3x_3 \\ x_3' = 4x_1 + 10x_2 + 5x_3 \end{cases}$

a) $A = \begin{pmatrix} 7 & 11 & 3 \\ -4 & -8 & -3 \\ 4 & 10 & 5 \end{pmatrix}$

Char. equation: $\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 11 & 3 \\ -4 & -8-\lambda & -3 \\ 4 & 10 & 5-\lambda \end{vmatrix} =$

$$= (7-\lambda)\left(\overbrace{(-8-\lambda)(5-\lambda)}^{(\lambda-5)(\lambda+8)} + 30\right) - 11\left(-4(5-\lambda) + 12\right) +$$

$$+3\left(\underbrace{-40 + 4(8+\lambda)}_{4\lambda-8}\right) = (7-\lambda)(\lambda^2 + 3\lambda \underbrace{-40+30}_{-10}) -$$

$$-11\left(\underbrace{4\lambda - 8}_{4(\lambda-2)}\right) + 3\left(\underbrace{4\lambda-8}_{4(\lambda-2)}\right) = (7-\lambda)(\lambda^2+3\lambda-10) -$$

$$-32(\lambda-2) = (*)$$

Rem As a matter of fact, since $\lambda^2+3\lambda-10 = (\lambda-2)(\lambda+5)$
it shows that $\lambda = 2$ is an eigenvalue so it is

not necessary to open all brackets but more simple is just to take out the factor $(\lambda-2)$. However, since such factorization usually is not seen on this stage and since I want to demonstrate another method, as we discussed in class, I will open the brackets here:

$$* = \underline{7\lambda^2} + \underline{\underline{21\lambda}} - 70 - \lambda^3 \underline{- 3\lambda^2} + \underline{\underline{10\lambda}} - \underline{\underline{32}}\lambda + 64 =$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \qquad (**)$$

If $\lambda = \frac{p}{q}$ is a solution, with integer p & q s.t. p, q are coprime then q divides 1 (the coefficient of λ^3) and p divides 6 $\Rightarrow$ all options are

$$\pm 1, \pm 2, \pm 3, \pm 6 \qquad (***)$$

We already know that $\lambda = 2$ is a root

(In general in this method one substitute numbers from $(***)$ instead of λ in $(**)$

$\lambda = 1$: $1 - 4 + 1 + 6 \neq 0$ $\quad \lambda = -1$ $\quad -1 - 4 + 1 + 6 = 0$ ✓

$\Rightarrow \lambda_1 = -1$

and then divide the characteristic polynomial by $\lambda+1$ to get a quadratic polynomial whose roots we can find by the quadratic formula, or we continue to substitute numbers from (**) into (*) in hope to get other integer roots and if not return to the method of division of polynomials (In our case we know that

$$\lambda_2 = 2 \overset{\text{Viete}}{\Longrightarrow} \text{ since } \lambda_1 \lambda_2 \lambda_3 = -\frac{\text{free term}}{\text{coefo of } \lambda_3} = -6$$

$$\text{so } (-1)\cdot 2\,\lambda_3 = -6 \Rightarrow \lambda_3 = 3$$

By the method of division we would make

$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 6 \\ \hline \lambda+1 & \lambda^3 - 4\lambda^2 + \lambda + 6 \\ & \underline{\lambda^3 + \lambda^2} \\ & \quad -5\lambda^2 + \lambda \\ & \quad \underline{-5\lambda^2 - 5\lambda} \\ & \qquad 6\lambda + 6 \end{array}$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$D = 25 - 24 = 1$$

$$\lambda_2 = \frac{5-1}{2} = 2$$

$$\lambda_3 = \frac{5+1}{2} = 3$$

Again, there are many ways to solve (**), I just gave some of them. So, the eigenvalues are

$$\boxed{\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 3}$$

11F) • An eigenvector for $\lambda = -1$

$$(A-(-1)I)v = (A+I)v = \begin{pmatrix} 8 & 11 & 3 \\ -4 & -7 & -3 \\ 4 & 10 & 6 \end{pmatrix} v = 0$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 8 & 11 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 4 & 10 & 6 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -4 & -7 & -3 & 0 \\ 8 & 11 & 3 & 0 \\ 4 & 10 & 6 & 0 \end{array}\right) \xrightarrow[\substack{R_3 \to R_3 + R_1 \\ R_1 \to -R_1}]{R_2 \to R_2 + 2R_1}$$

$$\left(\begin{array}{ccc|c} 4 & 7 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{3}R_2]{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 4 & 7 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

at least one zero rows as expected (actually also exactly one zero row in the case of distinct eigenvalues is expected)

$$\Rightarrow \begin{aligned} 4v_1 + 7v_2 + 3v_3 &= 0 \quad (\text{Eq } 1) \\ v_2 + v_3 &= 0 \quad (\text{Eq } 2) \end{aligned}$$

Take $v_3 = 1 \xRightarrow{(\text{Eq } 2)} v_2 = -1 \xRightarrow{\text{Eq } 1}$

$$4v_1 - 7 + 3 = 0 \Rightarrow v_1 = 1$$

so $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = -1 \Rightarrow e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a solution

• An eigenvector for $\lambda = 2$

12) $(A-2I)v = \begin{pmatrix} 5 & 11 & 3 \\ -4 & -10 & -3 \\ 4 & 10 & 3 \end{pmatrix} v = 0 \Rightarrow$

Augmented matrix is

$$\left(\begin{array}{ccc|c} 5 & 11 & 3 & 0 \\ -4 & -10 & -3 & 0 \\ 4 & 10 & 3 & 0 \end{array}\right) \underset{R_2 \to -R_2}{\overset{R_3 \to R_3 + R_1}{\sim}} \left(\begin{array}{ccc|c} 5 & 11 & 3 & 0 \\ 4 & 10 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_2 \to 5R_2 - 4R_1}{\sim}$$

$$\left(\begin{array}{ccc|c} 5 & 11 & 3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} 5v_1 + 11v_2 + 3v_3 = 0 & \text{(Eq 1)} \\ 2v_1 + v_3 = 0 & \text{(Eq 2)} \end{cases}$$

Take $v_2 = 1 \overset{\text{(Eq 2)}}{\Rightarrow} v_3 = -2 \overset{\text{(Eq 1)}}{\Rightarrow} 5v_1 + 11 - 6 = 0 \Rightarrow$

$v_1 = -1 \Rightarrow \quad v = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector of $\lambda = 2$

and $e^{2t}\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ is a solution

. An eigenvector for $\lambda = 3$

$(A-3I)v = \begin{pmatrix} 4 & 11 & 3 \\ -4 & -11 & -3 \\ 4 & 10 & 2 \end{pmatrix} v = 0 \Rightarrow$

Augmented matrix

$$\left(\begin{array}{ccc|c} 4 & 11 & 3 & 0 \\ -4 & -11 & -3 & 0 \\ 4 & 10 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 4 & 10 & 2 & 0 \\ -4 & -11 & -3 & 0 \\ 4 & 11 & 3 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_1]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 4 & 10 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \to -R_2]{R_1 \to \frac{1}{2}R_1}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{rl} 2v_1+5v_2+v_3=0 & \text{Eq 1} \\ v_2+v_3=0 & \text{Eq 2} \end{array}$$

Take $v_3=1 \overset{(\text{Eq}2)}{\Rightarrow} v_2=-1 \overset{(\text{Eq}1)}{\Rightarrow} 2v_1-5+1=0 \Rightarrow v_1=2 \Rightarrow$

$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_3=3 \Rightarrow e^{3t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is a solution

So, the general solution is

$$C_1 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + C_3 e^{3t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(b) $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \Rightarrow$

$$C_1\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + C_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ -1 & 1 & -1 & -2 \\ 1 & -2 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{array}{l} C_1-C_2+2C_3=3 \\ C_2+C_3=3 \\ C_3=1 \Rightarrow \end{array}$$

Backward substitution: $C_2=3-C_3=2 \Rightarrow$

14) $c_1 = 3 + c_2 - 2c_3 = 3 + 2 - 2 \cdot 1 = 3 \Rightarrow$

$$\boxed{x(t) = 3e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 2e^{2t}\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + e^{3t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}$$