Homework Assignment 7 in Differential Equations, MATH308

due March 28, 2012

Topics covered : *initial value problems with discontinuous and impulse forcing functions; convolution integrals (corresponds to sections 6.4, 6.5, 6.6 in the textbook)*

1. Find the solution of the initial value problem $y'' + 4y = g(t); \quad y(0) = 1,\, y'(0) = 3$, where

$$g(t) = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & t \ge 2\pi \end{cases}$$

2. Find the solution of the initial value problem $y'' + 5y' + 6y = g(t), \quad y(0) = 0,\, y'(0) = 2$, where

$$g(t) = \begin{cases} 0 & t < 1, \\ t & 1 \le t < 5, \\ 1 & t \ge 5. \end{cases}$$

3. Solve the initial value problem and sketch a graph of the solution:

$$y'' + y = -\delta(t - \pi) + \delta(t - 2\pi); \quad y(0) = 0,\, y'(0) = 1.$$

4. Solve the initial value problem $y'' + 6y'' + 5y = e^t \delta(t - 1); \quad y(0) = 0,\, y'(0) = 4.$

5. Use the convolution theorem to find the inverse Laplace transform of the given function:

   (a) $\dfrac{1}{s^3(s^2+1)}$; (b) $\dfrac{s}{(s^2+1)^2}$.

6. (a) Express the solution of the given initial value problem in terms of a convolution integral:

$$y'' - 2y' + 5y = g(t); \quad y(0) = 0,\, y'(0) = 2; \tag{1}$$

   (b) (bonus-15 points) Find the solution of the same initial value problem (1) using the method of variation of parameter. Show that your answer coincides with the answer obtained in item a) .

7. (bonus-15 points) (*on Laplace transform of periodic functions*) Recall that a function $f(t)$ is said to be *periodic of period* T if $f(t + T) = f(t)$ for all t. In this exercise you can use the following theorem: if $f(t)$ is periodic with period T and piecewise continuous on the interval $[0, T]$, then the Laplace transform $F(s)$ of $f(t)$ satisfies:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt \tag{2}$$

   (I posted the proof of this theorem in the class announcements of 03/21/2012). Graph the given periodic function $f(t)$ and find its Laplace transform based on formula (2) if

   (a) $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2, \end{cases}$ and $f(t)$ has period 2;

   (b) $f(t) = t, \quad 0 \le t < 1$, and $f(t)$ has period 1.