Homework #7 solutions – MATH 308-505 FALL 2013

1) a) $25y''-20y'+4y=0$

Char. eq. is $25r^2-20r+4=0 \Leftrightarrow (5r-2)^2=0 \Rightarrow r_1=r_2=\frac{2}{5}$ - repeated root

$\Downarrow$

Gen. solution is $y(t)=\boxed{C_1e^{\frac{2}{5}t}+C_2te^{\frac{2}{5}t}}=(C_1+C_2t)e^{\frac{2}{5}t}$

b) $y(0)=\alpha$, $y'(0)=5$

$y(0)=0 \Rightarrow C_1=\alpha$

$y'(t)=\frac{2}{5}C_1e^{\frac{2}{5}t}+C_2e^{\frac{2}{5}t}+\frac{2}{5}C_2te^{\frac{2}{5}t} \Rightarrow y'(0)=\frac{2}{5}\underbrace{C_1}_{\alpha}+C_2=5 \Rightarrow$

$$\frac{2}{5}\alpha+C_2=5 \Rightarrow C_2=5-\frac{2}{5}\alpha \Rightarrow$$

$$y(t)=\alpha e^{\frac{2}{5}t}+\left(5-\frac{2}{5}\alpha\right)te^{\frac{2}{5}t}=\left(\alpha+\left(5-\frac{2}{5}\alpha\right)t\right)e^{2/5t}$$

(c) The sign of $5-\frac{2}{5}\alpha$ is deciding here:

- If $5-\frac{2}{5}\alpha>0 \left(\Leftrightarrow \alpha<\frac{25}{2}\right) \Rightarrow y(t)\xrightarrow[t\to+\infty]{}+\infty$
- If $5-\frac{2}{5}\alpha=0 \left(\Leftrightarrow \alpha=\frac{25}{2}\right) \Rightarrow y(t)=\frac{25}{2}e^{2/5t}\xrightarrow[t\to+\infty]{}+\infty$
- If $5-\frac{2}{5}\alpha<0 \left(\Leftrightarrow \alpha>\frac{25}{2}\right) \Rightarrow y(t)\xrightarrow[t\to+\infty]{}-\infty$

So the conclusion is:
$$\boxed{\begin{array}{l} y(t)\to+\infty \Leftrightarrow \alpha\le\frac{25}{2} \\ y(t)\to-\infty \Leftrightarrow \alpha>\frac{25}{2}\end{array}}$$

2) $t^2y''+2ty'-6y=0$, $t>0$

$y_1(t)=t^2$ is a solution. Look for the second solution in the form

$-6\times$ $y(t)=v(t)t^2 \Rightarrow$

$2\times$ $y'(t)=2v(t)t+v'(t)t^2$

$1\times$ $y''(t)=2v(t)+2v'(t)t+2v'(t)t+v''t^2$

$$\left.\right\} \; t^2y''+2ty'-6y = \underbrace{(2+4-6)}_{0}t^2v+(4+2)t^3v'+t^4v''=0$$

$$\Rightarrow \quad 6t^3v' + t^4v'' = 0 \overset{\text{divide by } t^3}{\Leftrightarrow} tv'' + 6v' = 0$$

Let $w = v'$ $\Rightarrow$ we get the first order equation for w:

$$tw' + 6w = 0 \Leftrightarrow tw' = -6w \Leftrightarrow w' = -\frac{6}{t}w$$

$$w = \tilde{C}_1 e^{-\int \frac{6}{t} dt} = \tilde{C}_1 e^{-6\ln t} = \tilde{C}_1 t^{-6} \Rightarrow$$

$$v' = \tilde{C}_1 t^{-6} \Rightarrow v = \tilde{C}_1 \int t^{-6} dt + C_2 = \underbrace{\left(-\frac{\tilde{C}_1}{5}\right)}_{C_1} t^{-5} + C_2 =$$

$$= C_1 t^{-5} + C_2 \Rightarrow$$

$$y = v(t)t^2 = (C_1 t^{-5} + C_2)t^2 = C_1 \underbrace{t^{-3}}_{\text{second solution}} + C_2 t^2$$

$$\Downarrow$$

As the second (independent) solution one can take

$$y_2(t) = t^{-3}$$