# Homework # 8 Solutions FALL 2013

(a) (i) $7y''-20y'-3y=2013e^{-3t}$

Characteristic equation is

$$7r^2-20r-3=0$$

$$D/4=\left(\frac{b}{2}\right)^2-ac=100+21=121$$

$$r_{1,2}=\frac{-b/2\pm\sqrt{D/4}}{a}=\frac{10\pm11}{7}\ \begin{matrix}\nearrow 3\\ \searrow -\frac{1}{7}\end{matrix}$$

$\alpha=-3$ is not a root of characteristic polynomial $\Rightarrow$
the multiplicity is 0 $\Rightarrow$ $\boxed{y_p(t)=Ae^{-3t}}$ for some $A$

(ii) $7y''-20y'-3y=7e^{3t}$

The same char. eq., but $\alpha=3$ is a root (nonrepeated) $\Rightarrow$
$s=1$ $\Rightarrow$ $\boxed{y_p(t)=Ate^{3t}}$ for some $A$

(iii) $7y''-20y'-3y=7e^{3t}-5e^{-t/7}$

Both 3 and $-\frac{1}{7}$ are roots of characteristic polynomial
$\Rightarrow$ $y_p(t)=A_1te^{3t}+A_2te^{-t/7}$ for some $A_1$ and $A_2$

( In fact we consider 2 equations

$7y''-20y'-3y=7e^{3t}$ $\Rightarrow$ a particular solution is found in the form $A_1te^{3t}$ for some $A_1$

$7y''-20y'-3y=-5e^{-t/7}$ $\Rightarrow$ a particular solution is found in the form $A_2te^{-t/7}$ for some $A_2$

Then the sum of this two particular solutions is a solution of the equation: $\boxed{y_p(t)=A_1te^{3t}+A_2te^{-t/7}}$

(iv) $7y'' - 20y' - 3y = e^{3t}\sin t$

Here we have to check whether $3+i$ is a root of characteristic polynomial and it is not the case $\Rightarrow$ we look for a particular solution in the form

$$\boxed{A_1 e^{3t}\cos t + A_2 e^{3t}\sin t}$$ for some $A_1$ and $A_2$

(v) $7y'' - 20y' - 21y = 71(3t-1)e^{3t}$

In this case the char. equation is

$$7r^2 - 20r - 21 = 0$$

$$D/4 = (b/2)^2 - ac = 100 + 147 = 247$$

$$r_{1,2} = \frac{10 \pm \sqrt{247}}{2} \neq 3 \Rightarrow$$

$\lambda = 3$ is not a root and we look for a solution in the form $\boxed{(A_1 t + A_0)e^{3t}}$ for some $A_0$ and $A_1$

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<u>Rem</u> Actually I made a misprint in the formulation of this item: I ment the equation

$$7y'' - 20y' - 3y = 71(3t-1)e^{3t}$$

Then 3 is a "non-repeated" root of characteristic polynomial and $s=1$ $\Rightarrow$ we look for a solution in the form

$$y_p(t) = t(A_1 t + A_0)e^{3t}$$

(vi) $9y'' - 6y' + y = 51e^{2t}$

The characteristic equation is

$$9r^2 - 6r + 1 = 0 \Leftrightarrow (3r-1)^2 = 0 \Rightarrow r_{1,2} = \frac{1}{3}$$

$\alpha = 2$ is not a root $\Rightarrow$ we look for a solution in the

form $y(t) = \boxed{Ae^{2t}}$ for some $A$

(vii) $9y'' - 6y' + y = 2e^{t/3} - 2e^{t/3}\sin 2t$

The characteristic equation is the same as in the previous item, i.e. $r_{1,2} = \frac{1}{3}$

For $9y'' - 6y' + y = 2e^{t/3}$ $\alpha = \frac{1}{3}$ is a repeated root and we look for a solution in the form $At^2e^{t/3}$ for some $A$

For $9y'' - 6y' + y = -2e^{t/3}\sin 2t$ $\frac{1}{3} + 2i$ is not a root of characteristic equation $\Rightarrow$ we look for a solution in the form $B_1 e^{t/3}\cos 2t + B_2 e^{t/3}\sin 2t$

The sum of these 2 particular solutions is the solution of the original equation

$$\boxed{y_p(t) = At^2e^{t/3} + B_1 e^{t/3}\cos 2t + B_2 e^{t/3}\sin 2t}$$ for some $A$, $B_1$ and $B_2$

(VIII) $y'' + \omega_0^2 y = 2\cos\omega t + \sin\omega t$ where $\omega^2 \neq \omega_0^2$

Characteristic equation is $r^2 + \omega_0^2 = 0 \Rightarrow$

$r_{1,2} = \pm i\omega_0$

If $\omega^2 \neq \omega_0^2$ then $i\omega$ is not a root and we look for a particular solution in the form

$\boxed{y_p(t) = A_1 \cos\omega t + A_2 \sin\omega t}$ for some $A_1$ and $A_2$

(IX) $y'' + \omega_0^2 y = 2\cos\omega t + \sin\omega t$, $\omega^2 = \omega_0^2$

In this case $i\omega$ is a root and for $\omega \neq 0$ it is not a repeated root $\Rightarrow$ the multiplicity $s = 1$

and $\boxed{y_p(t) = t\,(A_1 \cos\omega t + A_2 \sin\omega t)}$

(Note that the case $\omega_0 = \omega = 0$ is also formally possible: in this case your equation is

$y'' = 2 \Rightarrow \alpha = 0$ has multiplicity 2 and you look for a solution in the form

$y(t) = At^2$)

(X) $18y'' + 30y' + 17y = e^{-5t/6}\left(7\cos\frac{t}{2} + 5\sin\frac{t}{2}\right)$

The characteristic equation is

$18r^2 + 30r + 17 = 0$

$D/4 = \left(\frac{b}{2}\right)^2 - ac = 15^2 - 18 \cdot 17 = 225 - 306 = -81 \Rightarrow$

$$r_{1,2} = \frac{-15 \pm 9i}{18} = -\frac{5}{6} \pm \frac{1}{2} i$$

Note that $\alpha + i\beta = -\frac{5}{6} + \frac{1}{2}i$ is a root of characteristic equation (non-repeated) $\Rightarrow$ the multiplicity $s = 1$, we look for a particular solution in the form

$$\boxed{y_p(t) = t e^{-\frac{5}{6}t}\left(A_1 \cos\frac{t}{2} + A_2 \sin\frac{t}{2}\right)}$$

(6) $7y'' - 20y' - 3y = e^{3t}\sin t$

By previous item $r_{1,2} = 3$ or $-\frac{1}{7}$

$\Downarrow$

gen. solution of the hom eq. is $C_1 e^{3t} + C_2 e^{-\frac{1}{7}t}$

$3+i$ is not a root $\Rightarrow$ a particular solution of the original nonhom. eq. is

$y(t) = e^{3t}(A_1 \cos t + A_2 \sin t)$

Find $A_1$ and $A_2$ by plugging into equation.

$-3 \quad y(t) = A_1 e^{3t}\cos t + A_2 e^{3t}\sin t$

$-20 \times \quad y'(t) = 3A_1 e^{3t}\cos t - A_1 e^{3t}\sin t + 3A_2 e^{3t}\sin t + A_2 e^{3t}\cos t =$

$= (3A_1 + A_2)e^{3t}\cos t + (-A_1 + 3A_2)e^{3t}\sin t$

$7 \times \quad y''(t) = (9A_1 + 3A_2)e^{3t}\cos t - (3A_1 + A_2)e^{3t}\sin t + (-3A_1 + 9A_2)e^{3t}\sin t +$

$+ (-A_1 + 3A_2)e^{3t}\cos t = (8A_1 + 6A_2)e^{3t}\cos t + (-6A_1 + 8A_2)e^{3t}\sin t$

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$7y'' - 20y' - 3y = (56A_1 + 42A_2 - 60A_1 - 20A_2 - 3A_1)e^{3t}\cos t + (-42A_1 + 56A_2 +$

$+ 20A_1 - 60A_2 - 3A_2)e^{3t}\sin t = (-7A_1 + 22A_2)e^{3t}\cos t + (-22A_1 - 7A_2)e^{3t}\sin t =$

$= e^{3t}\sin t \Rightarrow$

$-7A_1 + 22A_2 = 0 \Rightarrow A_2 = \frac{7}{22}A_1$

$-22A_1 - 7A_2 = 1 \Rightarrow -22A_1 - \frac{49}{22}A_1 = 1 \Rightarrow$

$-\frac{22^2 - 49}{22}A_1 = 1 \Rightarrow -\frac{533}{22}A_1 = 1 \Rightarrow A_1 = -\frac{22}{533} \Rightarrow A_2 = -\frac{7}{533}$

$\Rightarrow$ gen. solution is

$$\boxed{y(t) = -\frac{e^{3t}}{533}(7\cos t + 22\sin t) + C_1 e^{3t} + C_2 e^{-7t}}$$

(c) $y'' + \omega_0^2 y = 2\cos\omega t + \sin\omega t$

$\omega^2 \neq \omega_0^2 \Rightarrow$ a particular solution is of the form

$$\omega_0^2 \times \left| \begin{array}{l} y(t) = A_1 \cos\omega t + A_2 \sin\omega t \Rightarrow \\ y''(t) = -\omega^2 A_1 \cos\omega t - \omega^2 A_2 \sin\omega t \Rightarrow \end{array} \right.$$

$$y'' + \omega_0^2 y = (\omega_0^2 - \omega^2) A_1 \cos\omega t + (\omega_0^2 - \omega^2) A_2 \sin\omega t = 2\cos\omega t + \sin\omega t \Rightarrow$$

$$A_1 = \frac{2}{\omega_0^2 - \omega^2}, \quad A_2 = \frac{1}{\omega_0^2 - \omega^2}$$

$\Downarrow$

$$y_p(t) = \frac{2}{\omega_0^2 - \omega^2} \cos\omega t + \frac{1}{\omega_0^2 - \omega^2} \sin\omega t \Rightarrow$$

gen solution is

$$\boxed{y(t) = \frac{2}{\omega_0^2 - \omega^2} \cos\omega t + \frac{1}{\omega_0^2 - \omega^2} \sin\omega t + C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}$$

(d) $y'' + \omega_0^2 y = 2\cos\omega_0 t + \sin\omega_0 t \Rightarrow$

Since $\omega^2 = \omega_0^2$, $i\omega$ is a root of char polynomial

A particular solution can be found in the form

$$y(t) = A_1 t \cos\omega t + A_2 t \sin\omega t \Rightarrow$$

$$y'(t) = -A_1 \omega t \sin(\omega t) + A_2 \omega t \cos(\omega t) + A_1 \cos\omega t + A_2 \sin\omega t$$

$$y''(t) = -A_1 \omega^2 t \sin\omega t - A_2 \omega^2 t \sin\omega t - 2A_1 \omega \sin\omega t + 2A_2 \omega \cos\omega t$$

$$y''(t) + \omega_0^2 y(t) \overset{\text{because } \omega_0^2 = \omega^2}{=} -2A_1 \omega \sin\omega t + 2A_2 \omega \cos\omega t = 2\cos\omega t + \sin\omega t$$

$$-2A_1 \omega = 1, \quad 2A_2 \omega = 2$$

$\Downarrow$

$$A_1 = -\frac{1}{2\omega}, \quad A_2 = \frac{1}{\omega} \Rightarrow$$

Gen. solution is $y(t) = \boxed{t\left(-\frac{1}{2\omega}\cos\omega t + \frac{1}{\omega}\sin\omega t\right) + C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}$

(here $\omega^2 = \omega_0^2$)