Homework assignment #9, Solutions, MATH 308-505, FALL 2013

1(a) $3y''-6y'+6y=e^x\sec x \iff y''-2y'+2y=\underbrace{\frac{1}{3}e^x\sec x}_{g(x)}$

Solution: • Homogeneous equation:

$$3y''-6y'+6y=0 \iff$$

$$y''-2y'+2y=0$$

• The characteristic equation is

$$r^2-2r+2=0$$

$$D/4=1-2=-1 \Rightarrow$$

$$r_{1,2}=1\pm i$$

• The gen. solution of homogeneous equation is

$$y(x)=C_1e^x\cos x+C_2e^x\sin x$$

$$\Downarrow$$

We look for a solution of nonhom. eq. in the form

$$y(x)=u_1(x)e^{-x}\cos x+u_2(x)e^{-x}\sin x$$

such that

$$\begin{cases} u_1'(x)e^x\cos x+u_2'(x)e^x\sin x=0 & (Eq 1)\\ u_1'(x)(-e^x\cos x-e\sin x)+u_2'(x)(-e^x\sin x+e^x\cos x)=\frac{1}{3}e^x\sec x & (Eq 2)\\ -u_1'(x)e^x\sin x+u_2'(x)e^x\cos x=\frac{1}{3}e^x\sec x & (Eq 2')\end{cases}$$

(Eq 1)+(Eq 2) $\Rightarrow$

• $\cos x$ (Eq 1) $-$ $\sin x$ (Eq 2') $\Rightarrow u_1'(x)\cancel{e^x}=-\frac{1}{3}\cancel{e^x}\tan x \Rightarrow u_1'(x)=-\frac{1}{3}\tan x$

$$\Rightarrow u_1(x)=-\frac{1}{3}\int\frac{\sin x}{\cos x}dx+C_1=\frac{1}{3}\int\frac{du}{u}+C_1=\frac{1}{3}\ln|u|+C_1=\frac{1}{3}\ln|\cos x|+C_1$$

$u=\cos x$

$du=-\sin x\,dx$

• $\sin x$ (Eq 1) $+$ $\cos x$ (Eq 2') $\Rightarrow u_2'(x)\cancel{e^x}=\frac{1}{3}\cancel{e^x} \Rightarrow u_2'(x)=\frac{1}{3} \Rightarrow$

$$u_2(x)=\frac{1}{3}x+C_2 \Rightarrow$$

$$\boxed{y(x)=\left(\frac{1}{3}\ln|\cos x|+C_1\right)e^x\cos x+\left(\frac{1}{3}x+C_2\right)e^x\sin x=\frac{1}{3}e^x\ln|\cos x|\cos x+\frac{1}{3}xe^x\sin x+C_1e^x\cos x+C_2e^x\sin x}$$

(6) $y'' - 4y' + 4y = \underbrace{(x^2+1)e^{2x}}_{g(x)}$

$y(0) = 1,$

$y'(0) = 0$

Solution. Homogeneous equation

$$y'' - 4y' + 4 = 0$$

- The characteristic equation is

$$r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Rightarrow r = 2 \text{ is a repeated root}$$

- The general solution of the homogeneous equation is

$$y(x) = C_1 e^{2x} + C_2 x e^{2x}$$

- We look for a solution of non-homog. eq. in the form

$$y(x) = u_1(x) e^{2x} + u_2(x) x e^{2x}$$

such that
$$\begin{cases} u_1'(x) e^{2x} + u_2'(x) x e^{2x} = 0 & (Eq\,1) \\ u_1'(x) 2e^{2x} + u_2'(x)(e^{2x} + 2x e^{2x}) = (x^2+1)e^{2x} & (Eq\,2) \end{cases}$$

$(Eq\,2) - 2(Eq\,1) \Rightarrow \quad u_2'(x)\cancel{e^{2x}} = (x^2+1)\cancel{e^{2x}} \Rightarrow u_2'(x) = x^2 + 1 \Rightarrow u_2(x) = \frac{1}{3}x^3 + x + C_2$

Substitute to $(Eq\,1)$: $u_1'(x) + u_2'(x)\, x = 0 \Rightarrow u_1'(x) = -u_2'(x) x = -x^3 - x \Rightarrow$

$$u_1(x) = -\frac{1}{4}x^4 - \frac{1}{2}x^2 + C_1 \Rightarrow$$

$$y(x) = \left(-\frac{1}{4}x^4 - \frac{1}{2}x^2 + C_1\right)e^{2x} + \left(\frac{1}{3}x^3 + x + C_2\right)x e^{2x} =$$

$$= \left(-\frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{3}x^4 + x^2\right)e^{2x} + C_1 e^{2x} + C_2 x e^{2x} =$$

$$= \boxed{\left(\frac{1}{12}x^4 + \frac{1}{2}x^2 + C_2 x + C_1\right)e^{2x}}$$

Determine C_1 and C_2 from the initial conditions

$y(0) = 1 \Rightarrow C_1 = 1$

$y'(0) = 0$: $y'(x) = \left(\frac{1}{3}x^3 + x + C_2\right)e^{2x} + 2\left(\frac{1}{12}x^4 + \frac{1}{2}x^2 + C_2 x + 1\right)e^{2x} \Rightarrow$

$$y'(0) = C_2 + 2 = 0 \Rightarrow C_2 = -2 \Rightarrow$$

$$\boxed{y(x) = \left(\frac{1}{12}x^4 + \frac{1}{2}x^2 - 2x + 1\right)e^{2x}}$$

(c) Given: $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2} \Rightarrow y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = x^{-1/2}$

$y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ constitute the fundamental set of solution of the homogeneous equation $\Rightarrow$

A solution of nonhomogeneous equation can be found in the form

$y(x) = u_1(x)x^{-1/2}\cos x + u_2(x)x^{-1/2}\sin x$, where

$$\begin{cases} u_1'(x)x^{-1/2}\cos x + u_2'(x)x^{-1/2}\sin x = 0 & \text{(Eq 1)} \\ u_1'(x)\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right) + u_2'(x)\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right) = x^{-1/2} & \text{(Eq 2)} \end{cases}$$

$(Eq\ 2) + \frac{1}{2}(Eq\ 1) \Rightarrow -u_1'(x)x^{-1/2}\sin x + u_2'(x)x^{-1/2}\cos x = x^{-1/2}$

So we have
$$\begin{cases} \cos x\, u_1'(x) + \sin x\, u_2'(x) = 0 & \text{(Eq 3)} \\ -\sin x\, u_1'(x) + \cos x\, u_2'(x) = 1 & \text{(Eq 4)} \end{cases}$$

$\cos x(Eq\ 3) - \sin x(Eq\ 4) \Rightarrow u_1'(x) = -\sin x \Rightarrow u_1(x) = \cos x + C_1$

$\sin x(Eq\ 3) + \cos x(Eq\ 4) \Rightarrow u_2'(x) = \cos x \Rightarrow u_2(x) = \sin x + C_2$

$\Rightarrow y(x) = (\cos x + C_1)x^{-1/2}\cos x + (\sin x + C_2)x^{-1/2}\sin x =$

$= x^{-1/2}\underbrace{(\cos^2 x + \sin^2 x)}_{1} + C_1x^{-1/2}\cos x + C_2x^{-1/2}\sin x =$

$= \boxed{x^{-1/2} + C_1x^{-1/2}\cos x + C_2x^{-1/2}\sin x}$