Homework #9 MATH 308 Solutions

Problem 1 $x_1' = x_2 + x_3$
$x_2' = x_1 + x_3$
$x_3' = x_1 + x_2$

1) The matrix of the system is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

2) The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) =$$

$$= -\lambda(\lambda-1)(\lambda+1) + 2(\lambda+1) = -(\lambda+1)\underbrace{(\lambda^2 - \lambda - 2)}_{(\lambda+1)(\lambda-2)} = -(\lambda+1)^2(\lambda-2) = 0$$

Two eigenvalues $\lambda = -1$ and $\lambda = 2$. The multiplicity of $\lambda = -1$ in the characteristic polynomial is equal to 2, i.e $\lambda = -1$ is a repeated eigenvalue

3) Find the eigenspace corresponding to $\lambda = -1$

$$(A - (-1)I)V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \underbrace{V_1 + V_2 + V_3 = 0}_{\text{plane, the eigenspace of } \lambda = -1}$$

Find a basis in this eigenspace.

For this set, for example, $V_2 = 1, V_3 = 0 \Rightarrow V_1 = -1$

$V_2 = 0, V_3 = 1 \Rightarrow V_1 = -1$

So as a basis in this eigenspace one can take $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Then the corresponding independent solutions are $e^{-t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

4) Find the eigenvector corresponding to $\lambda = 2$

$$(A-2I)v = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow[R_3 \ominus 2R_2 + R_1]{R_2 \ominus 2R_2 + R_1} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \xrightarrow[R_3 \ominus R_3 + R_2]{R_2 \ominus \frac{1}{3} R_2} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $-2v_1 + v_2 + v_3 = 0$
$-v_2 + v_3 = 0$

Set $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow -2v_1 + 2 = 0 \Rightarrow v_1 = 1 \Rightarrow$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector, corresponding to $\lambda = 2$

This gives one more solution $e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$

The general solution is

$$x(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Determine the constants c_1, c_2 and c_3 s.t. $x(t)$ satisfies the given initial conditions.

$$x(0) = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$$

Let us use the Gauss elimination method

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_2+R_1} \left(\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3+R_2}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right) \to \begin{array}{l} -c_1 - c_2 + c_3 = -1 \quad -c_1 + 1 + 1 = -1 \Rightarrow \boxed{c_1 = 3} \\ -c_2 + 2c_3 = 3 \quad -c_2 + 2 = 3 \Rightarrow \boxed{c_2 = -1} \\ 3c_3 = 3 \Rightarrow \boxed{c_3 = 1} \end{array}$$

Therefore the solution of the given IVP is

$$3e^{-t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 2 a)

$$\begin{cases} x_1' = 6x_1 - x_2 \\ x_2' = 5x_1 + 2x_2 \end{cases}$$

1) The matrix of the system is $A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$

2) The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - \operatorname{tr} A \lambda + \det A =$$

$$= \lambda^2 - 8\lambda + 17 = (\lambda - 4)^2 + 1 = 0 \Rightarrow$$

$\lambda_{1;2} = 4 \pm i$

Equivalently

$D = 64 - 4 \cdot 17 = 64 - 68 = -4$

$\lambda_{1,2} = \dfrac{8 \pm \sqrt{-4}}{2} = 4 \pm i$

3) Find the eigenvector corresponding to the eigenvalue $\lambda = 4+i$

$$\left(A-(4+i)I\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{cases} (2-i)v_1 - v_2 = 0 \Rightarrow v_2 = (2-i)v_1 . \text{ Set } v_1 = 1 \Rightarrow v_2 = (2-i) \Longrightarrow \\ 5v_1 - (2+i)v_2 = 0 \end{cases}$$

(note that the second equation is obtained from the first one by multiplication by $2+i$)

$\Rightarrow \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 4+i$

It gives a complex-valued solution of our system

$$e^{(4+i)t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

4) Find the real and the imaginary parts of $e^{(4+i)t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

This will give us the fundamental set of <u>real-valued</u> solutions of our system

$$e^{(4+i)t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} = e^{4t}(\cos t + i\sin t)\begin{pmatrix} 1 \\ 2-i \end{pmatrix} =$$

$$= e^{4t}\begin{pmatrix} \cos t + i\sin t \\ 2\cos t + \sin t + i(2\sin t - \cos t) \end{pmatrix} =$$

$$= e^{4t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i e^{4t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

$$\Rightarrow \quad \mathrm{Re}\left(e^{(4+i)t}\begin{pmatrix}1\\2-i\end{pmatrix}\right) = e^{4t}\begin{pmatrix}\cos t\\2\cos t+\sin t\end{pmatrix}$$

$$\mathrm{Im}\left(e^{(4+i)t}\begin{pmatrix}1\\2-i\end{pmatrix}\right) = e^{4t}\begin{pmatrix}\sin t\\2\sin t-\cos t\end{pmatrix}$$

$$\Downarrow$$

Fundamendal matrix can be taken as

$$\begin{pmatrix} e^{4t}\cos t & e^{4t}\sin t \\ e^{4t}(2\cos t+\sin t) & e^{4t}(2\sin t-\cos t)\end{pmatrix}$$

General solution is

$$\boxed{x(t) = C_1 e^{4t}\begin{pmatrix}\cos t\\2\cos t+\sin t\end{pmatrix} + C_2 e^{4t}\begin{pmatrix}\sin t\\2\sin t-\cos t\end{pmatrix}}$$

Problem 2 b) $\quad A = \begin{pmatrix}1 & -1 & 2\\-1 & 1 & 0\\-1 & 0 & 1\end{pmatrix}$

1. Find the eigenvalues
The characteristic equation is

$$\det(A-\lambda I) = \begin{vmatrix}1-\lambda & -1 & 2\\-1 & 1-\lambda & 0\\-1 & 0 & 1-\lambda\end{vmatrix} = (1-\lambda)^3-(1-\lambda)+2(1-\lambda) =$$

$$= (1-\lambda)\left((\lambda-1)^2+1\right) = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_{2,3} = 1\pm i$$

2. Find an eigenvector corresponding to $\lambda = 1$

$$(A-I)v = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\left.\begin{array}{l} -v_2 + 2v_3 = 0 \quad . \text{ Set } v_3 = 1 \Rightarrow v_2 = 2 \\ v_1 = 0 \end{array}\right\} \Rightarrow$$

$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1 \Rightarrow e^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is a solution of our system

3. Find the eigenvector, corresponding to the eigenvalue $\lambda_2 = 1+i$

$$\left(A - (1+i)I\right)v = \begin{pmatrix} -i & -1 & 2 \\ -1 & -i & 0 \\ -1 & 0 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By Gauss elimination:

$$\left(\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow iR_3 - R_1]{R_2 \leftrightarrow iR_2 - R_1} \left(\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow R_3 - \frac{1}{2}R_2]{R_2 \leftrightarrow \frac{1}{2}R_2}$$

$$\rightarrow \left(\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} -iv_1 - v_2 + 2v_3 = 0 \\ v_2 - v_3 = 0 \end{array}$$

Set $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow -iv_1 - 1 + 2 = 0 \Rightarrow$
$-iv_1 = -1 \Rightarrow v_1 = \frac{1}{i} = -i$

$\Rightarrow \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 1+i \Rightarrow$

$e^{(1+i)t} \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}$ is a complex-value solution

4. Find the real and the imaginary parts of

$$e^{(1+i)t}\begin{pmatrix}-i\\1\\1\end{pmatrix}:$$

$$e^{(1+i)t}\begin{pmatrix}-i\\1\\1\end{pmatrix} = e^{t}(\cos t + i\sin t)\begin{pmatrix}-i\\1\\1\end{pmatrix} =$$

$$= e^{t}\begin{pmatrix}\sin t - i\cos t\\ \cos t + i\sin t\\ \cos t + i\sin t\end{pmatrix} = e^{t}\begin{pmatrix}\sin t\\ \cos t\\ \cos t\end{pmatrix} + i e^{t}\begin{pmatrix}-\cos t\\ \sin t\\ \sin t\end{pmatrix}$$

$\Downarrow$ (combining 2. and 4)

The fundamendal set of solutions is

$$e^{t}\begin{pmatrix}0\\2\\1\end{pmatrix},\quad e^{t}\begin{pmatrix}\sin t\\ \cos t\\ \cos t\end{pmatrix},\quad e^{t}\begin{pmatrix}-\cos t\\ \sin t\\ \sin t\end{pmatrix}$$

The fundamendal matrix is

$$\begin{pmatrix}0 & e^{t}\sin t & -e^{t}\cos t\\ 2e^{t} & e^{t}\cos t & e^{t}\sin t\\ e^{t} & e^{t}\cos t & e^{t}\sin t\end{pmatrix}$$

The general solution is

$$\boxed{X(t) = C_1 e^{t}\begin{pmatrix}0\\2\\1\end{pmatrix} + C_2 e^{t}\begin{pmatrix}\sin t\\ \cos t\\ \cos t\end{pmatrix} + C_3 e^{t}\begin{pmatrix}-\cos t\\ \sin t\\ \sin t\end{pmatrix}}$$

Problem 3 a)

$$A = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix}$$

1) Find the eigenvalues of A

The characteristic equation is

$$\det |A - \lambda I| = \lambda^2 - \operatorname{tr} A \, \lambda + \det A = \lambda^2 + 4\lambda + 5 =$$

$$= (\lambda+2)^2 + 1 = 0 \Rightarrow \lambda_{1,2} = -2 \pm i$$

Equivalently
$\lambda^2 + 4\lambda + 5 = 0$
$D = 16 - 20 = -4$
$\lambda_{1,2} = \dfrac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$

2) Find the eigenvector corresponding to $\lambda = -2+i$

$$\left(A - (-2+i)I\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \left(A + (2-i)I\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1+i & -1 \\ 2 & 1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(-1-i)v_1 - v_2 = 0 \Leftrightarrow (1+i)v_1 + v_2 = 0$ set $v_1 = 1 \Rightarrow v_2 = -1-i \Rightarrow$

$2v_1 + (1-i)v_2 = 0$ (note that the second equation is the first one multiplied by $(-1+i)$)

$\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2+i \Rightarrow$

$e^{(-2+i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ is a complex-valued solution

Find its real and imaginary part

$$e^{(-2+i)t}\begin{pmatrix}1\\-1-i\end{pmatrix} = e^{-2t}(\cos t + i\sin t)\begin{pmatrix}1\\-1-i\end{pmatrix} =$$

$$= e^{-2t}\begin{pmatrix}\cos t + i\sin t\\ -\cos t + \sin t + i(-\sin t - \cos t)\end{pmatrix} =$$

$$= e^{-2t}\begin{pmatrix}\cos t\\ \sin t - \cos t\end{pmatrix} + ie^{2t}\begin{pmatrix}\sin t\\ -\sin t - \cos t\end{pmatrix} \Rightarrow$$

The general solution is

$$x(t) = C_1 e^{-2t}\begin{pmatrix}\cos t\\ \sin t - \cos t\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}\sin t\\ -\sin t - \cos t\end{pmatrix}$$

If $x(0) = \begin{pmatrix}-1\\1\end{pmatrix}$ then

$$C_1\begin{pmatrix}1\\-1\end{pmatrix} + C_2\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \Leftrightarrow$$

$$\boxed{C_1 = -1}$$

$$-C_1 - C_2 = 1 \Rightarrow 1 - C_2 = 1 \Rightarrow \boxed{C_2 = 0} \Rightarrow$$

$$\boxed{x(t) = e^{-2t}\begin{pmatrix}\cos t\\ \sin t - \cos t\end{pmatrix}}$$

Problem 3 B)

$$A = \begin{pmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix}$$

1. Find the eigenvalues of A

The characteristic equation

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -12 & -14 \\ 1 & 2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left((2-\lambda)(-2-\lambda)+3\right)+12(-2-\lambda+3)-14(1-2+\lambda) =$$

$$= (1-\lambda)\left(\underbrace{(\lambda-2)(\lambda+2)+3}_{\lambda^2-4+3}\right)-12(\lambda-1)-14(\lambda-1) =$$

$$= -(\lambda-1)(\lambda^2-1+26) = -(\lambda-1)(\lambda^2+25) = 0 \Rightarrow$$

$$\lambda_{1,2} = \pm 5i, \quad \lambda_3 = 1$$

2. Find an eigenvector corresponding to $\lambda = 5i$

$$(A-5iI)v = \begin{pmatrix} 1-5i & -12 & -14 \\ 1 & 2-5i & -3 \\ 1 & 1 & -2-5i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Use the Gauss elimination

$$\left(\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 1 & 2-5i & -3 & 0 \\ 1 & 1 & -2-5i & 0 \end{array}\right) \xrightarrow[R_3=(1-5i)R_3-R_1]{R_2\leftarrow(1-5i)R_2-R_1} \left(\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & (1-5i)(2-5i)+12 & -3+15i+14 & 0 \\ 0 & (1-5i)+12 & -(1-5i)(2+5i)+14 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & 2-25-10i-5i+12 & 11+15i & 0 \\ 0 & 13-5i & -2-25+10i-5i+14 & 0 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & -11-15i & 11+15i & 0 \\ 0 & 13-5i & -13+5i & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow \frac{R_3}{13-5i}]{R_2 \leftrightarrow \frac{R_2}{11+15i}}$$

$$\left(\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3+R_2} \left(\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad (1-5i)v_1 - 12v_2 - 14v_3 = 0$$
$$-v_2 + v_3 = 0$$

Set $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow (1-5i)v_1 - 12 - 14 = 0 \Rightarrow (1-5i)v_1 = 26 \Rightarrow$

$$v_1 = \frac{26}{1-5i} = \frac{\cancel{26}(1+5i)}{\cancel{1^2+5^2}} = 1+5i \quad \Rightarrow$$

$\begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5i$

$\Rightarrow e^{5it}\begin{pmatrix}1+5i\\1\\1\end{pmatrix}$ is a complex-value solution

Find its real and imaginary part

$$e^{5it}\begin{pmatrix}1+5i\\1\\1\end{pmatrix} = \left(\cos 5t + i\sin 5t\right)\begin{pmatrix}1+5i\\1\\1\end{pmatrix} =$$

$$= \begin{pmatrix}\cos 5t - 5\sin 5t + i\left(\sin 5t + 5\cos 5t\right)\\ \cos 5t + i\sin 5t\\ \cos 5t + i\sin 5t\end{pmatrix} \Rightarrow$$

$\begin{pmatrix}\cos 5t - 5\sin 5t\\ \cos 5t\\ \cos 5t\end{pmatrix}$ and $\begin{pmatrix}\sin 5t + 5\cos 5t\\ \sin 5t\\ \sin 5t\end{pmatrix}$ are solutions

of our system

3. Find an eigenvector corresponding to $\lambda = 1$

$$(A-I)v = \begin{pmatrix}0 & -12 & -14\\ 1 & 1 & -3\\ 1 & 1 & -3\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$ $\rightarrow$ the second and third equations coinside

$-12v_2 - 14v_3 = 0 \Leftrightarrow -6v_2 + 7v_3 = 0$, Set $v_3 = 6 \Rightarrow v_2 = -7 \Rightarrow$

$v_1 + v_2 - 3v_3 = 0$

$\Rightarrow v_1 - 7 - 3\cdot 6 = v_1 - 25 = 0 \Rightarrow v_1 = 25$

$\Rightarrow \begin{pmatrix}25\\-7\\6\end{pmatrix}$ is an eigenvector corresponding to $\lambda = 1$

$\Rightarrow$ The general solution is

$$x(t) = C_1 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + C_2 \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} + C_3 e^t \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}$$

If $x(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$ then

$$C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} 1 & 5 & 25 \\ 1 & 0 & -7 \\ 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

By Gauss elimination.

$$\left(\begin{array}{ccc|c} 1 & 5 & 25 & 4 \\ 1 & 0 & -7 & 6 \\ 1 & 0 & 6 & -7 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 5 & 25 & 4 \\ 1 & 0 & -7 & 6 \\ 0 & 0 & 13 & -13 \end{array}\right) \Rightarrow$$

$C_1 + 5C_2 + 25C_3 = 4 \qquad -1 + 5C_2 - 25 = 4 \Rightarrow 5C_2 = 30 \Rightarrow C_2 = 6$

$C_1 - 7C_3 = 6 \qquad C_1 + 7 = 6 \Rightarrow C_1 = -1$

$13C_3 = -13 \Rightarrow C_3 = -1$

So

$$\boxed{x(t) = -\begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} - e^t \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}}$$

Problem 4

(a) Eigenvalues of A

The characteristic equation

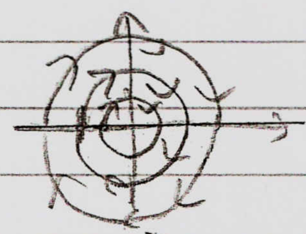
$$\det|A-\lambda I| = \lambda^2 - \mathrm{tr}A\,\lambda + \det A = \lambda^2 - 2\alpha\lambda + \alpha^2+1 =$$
$$= (\lambda-\alpha)^2+1 = 0 \Rightarrow \boxed{\lambda_{1,2} = \alpha \pm i}$$

(b) $\alpha = 0$: if $\alpha < 0$ the origin is <u>a spiral sink</u>

if $\alpha = 0$ the origin is <u>a center</u>

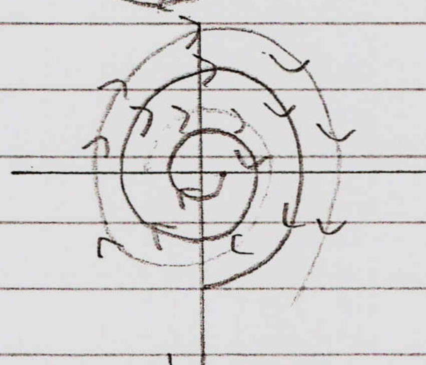
if $\alpha > 0$ the origin is <u>a spiral source</u>

(c) $\alpha = 0$



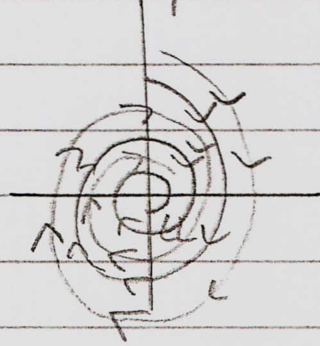
$\rightarrow$ center

$\alpha = 1$



spiral source

$\alpha = -1$



Spiral sink