Series of Solutions near an Ordinary Point (sections 5.2 and 5.3)

We consider differential equations of the type

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

DEFINITION 1. *A point x_0 is called an* **ordinary point** *of differential equation* (1) *if the functions* $p(x) := \dfrac{Q(x)}{P(x)}$ *and* $q(x) := \dfrac{R(x)}{P(x)}$ *are analytic at x_0, maybe after defining $p(x_0)$ and $q(x_0)$ appropriately (i.e. by continuity). Otherwise, the point x_0 is called a* **singular point** *of differential equation* (1).

EXAMPLE 2. *If the functions $P(x)$, $Q(x)$, and $R(x)$ are analytic and $P(x_0) \neq 0$, then x_0 is an ordinary point of* (1).

REMARK 3. *In general, if $P(x_0) = 0$ it does not mean that x_0 is a singular point of* (1)*, because it might be that also $Q(x_0) = 0$ and $R(x_0) = 0$ and the functions* $\dfrac{Q(x)}{P(x)}, \dfrac{R(x)}{P(x)}$ *might be defined at x_0 by continuity (namely* $\lim\limits_{x\to x_0} \dfrac{Q(x)}{P(x)}$ *and* $\lim\limits_{x\to x_0} \dfrac{R(x)}{P(x)}$ *may exist and be finite).*

EXAMPLE 4. *Given*

$$(1 - x^2)y'' + (x^2 + x - 2)y'' + (x^3 - 1)y = 0$$

a) is $x_0 = 1$ ordinary or singular?

b) is $x_0 = -1$ ordinary or singular?

EXAMPLE 5. *Given*

$$\sin^2 x\, y'' + x^2 y' + (1 - \cos x) y = 0$$

a) is $x_0 = 0$ ordinary or singular?

b) is $x_0 = 2\pi$ ordinary or singular?

THEOREM 6. *1. If x_0 is an ordinary point of differential equation (1), then any solution $y(x)$ of (1) is analytic at $x = x_0$, i.e can be found as a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$.*

2. The radius of convergence of this series is at least as large as the minimum of the radii of convergence of the Taylor series at x_0 of functions $p(x) := \dfrac{Q(x)}{P(x)}$ and $q(x) := \dfrac{R(x)}{P(x)}$.

REMARK 7. *The coefficients a_n of the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ with $n \geq 2$ are uniquely determined by the first two coefficients a_0 and a_1. Moreover, a_n are expressed linearly in terms of a_0 and a_1 so that*

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where $y_1(x)$ is the solution satisfying the initial conditions $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x)$ is the solution satisfying the initial conditions $y_2(x_0) = 0$, $y_2'(x_0) = 1$.

EXAMPLE 8. *Given differential equation*

$$xy'' + y' + xy = 0.$$

a) Seek power series solutions of this equation about $x_0 = 1$: find the recurrence relation for coefficients of the power series about $x_0 = 1$ representing a solution (in general, a recurrence relation is a relation expressing the nth coefficients a_n in terms of some previous ones).

b) Find the first five terms in the power expansion about $x_0 = 1$ of the solution of the equation (1) satisfying initial conditions $y(1) = 3$, $y'(1) = 1$.

Radius of convergence as the minimal distance to singularities.

There is a more elegant way to find the radius of convergence of the Taylor series at x_0 of the analytic function f rather than calculating the coefficients of this Taylor series (i.e. derivatives of any order of f at x_0). For this we pass to the complex plane.

Assume for simplicity that $f(x) = \frac{Q}{P}$, where Q and P are polynomials and all common linear factors (in general with complex coefficients) of Q and P are canceled. Then f is analytic at x_0 if and only if $P(x_0) \neq 0$ and **the radius of convergence of the Taylor series about x_0 is equal to the distance to the nearest complex zero of P.**

EXAMPLE 9. *Given $f(x) = \frac{1}{1+x^2}$ what is the radius of convergence of the Taylor series of f*

a) around $x_0 = 0$

b) around $x_0 = 1$

c) around $x_0 = 2$

REMARK 10. *Note that the function $f(x) = \frac{1}{1+x^2}$ of the previous example is defined for all real x, so we see the "problem" (non-convergence of Taylor series for $|x| > 1$) only when passing to the complex plane. This is another example when the use of complex numbers is very natural and helpful.*

EXAMPLE 11. *Determine a lower bound for the radius of convergence of series solutions about each given point of the following equation:*

a) $xy'' + y' + xy = 0$ about $x_0 = 1$ (as in Example 8)

b) $(x^2 + 2x + 2)y'' + xy' + 4y = 0$ about

i) $x_0 = 0$

ii) $x_0 = -1$

iii) $x_0 = -3$