# Applied Calculus on the Web

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# Chapter 1 Polynomials and Modeling

## 1.1 Linear Functions

Recall that a line is a function of the form $y = mx + b$, where $m$ is the slope of the line (how steep the line is) and $b$ gives the $y$-intercept (where the line crosses the $y$-axis). Using the notation given in Chapter 0, a line is a **linear function** in the form $f(x) = mx + b$.

**Example 1.1** Find the intercepts and slope of the linear function $f(x) = \frac{3}{4}x - \frac{15}{4}$.

**Solution** To find the $y$-intercept, you set $x = 0$. Therefore, we have

$$f(0) = \frac{3}{4}(0) - \frac{15}{4} = -\frac{15}{4}$$

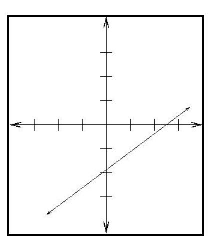
Thus we say that the $y$-intercept is the ordered pair $(0, -\frac{15}{4})$. Set $y = f(x) = 0$ to find the $x$-intercept,

$$\begin{aligned} f(x) = \frac{3}{4}x - \frac{15}{4} &= 0 \\ 3x - 15 &= 0 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

So, the $x$-intercept is the ordered pair (5, 0). ◈

In this form the slope is the constant in front of the $x$ and, therefore is equal to ¾. The intercepts of this function can be seen in Figure 1.1.

**Figure 1.1** Graph of Example 1.1



**Example 1.2** Given $f(x) = -\frac{2}{5}x + 3$, if $x$ increases by 10, what is the corresponding change in $f(x)$?

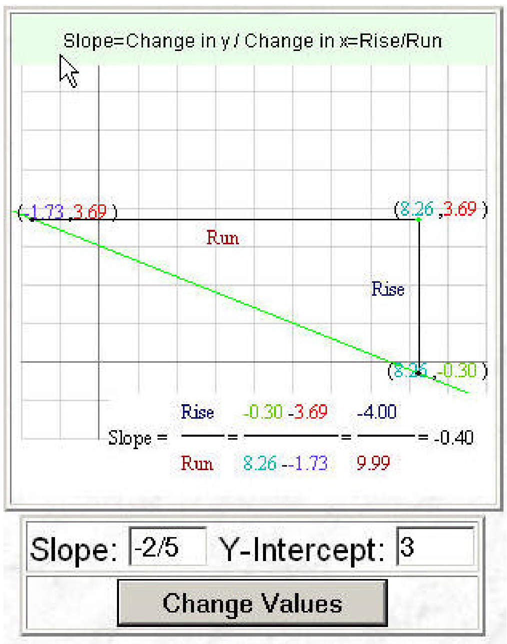
**Solution** The slope of a line is the amount of change in y when x increases by one unit. Since $m = \Delta y / \Delta x = {}^{-2}/_{5} = -0.4$, therefore, if x increases by 1, y decreases by 0.4 units.

Now, we have ? x = 10, then

$$\begin{aligned} \Delta y &= m \cdot \Delta x \\ &= -0.4 \cdot 10 \\ &= -4 \end{aligned}$$

Using the Lines and Slope Applet, we adjust the change in $x$ (the "Run") to be about 10 and look for the corresponding change in the function value (the "Rise"). From the screen we see this to be −4.

**Figure 1.2** Applet for Example 1.2




BUSINESS APPLICATIONS

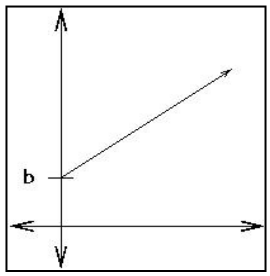
The aspects of linear functions we just explored have particular meaning when discussing real life applications of linear functions. One application of linear functions can be seen in business when discussing linear cost, revenue, and profit.

A **linear cost function** consists of two parts: variable costs (costs that depend on the number of items produced) and fixed costs (costs independent of the number of items produced). The total cost function is the sum of these costs.

$$\begin{aligned} \text{Total Costs } &= \text{Variable Costs + Fixed Costs} \\ C(x) &= mx + b \end{aligned}$$

where $x$ represents the number of items produced, $m$ represents the cost to produce each item, and $b$ represents the fixed costs.

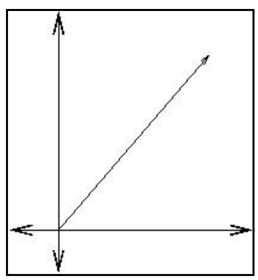
**Figure 1.3** Graph of a Generic Cost Function




A **linear revenue function** is found by multiplying the selling price of each item sold, $p$, by the total number of items sold, $x$. Thus, we have

$$\begin{aligned} \text{Revenue } &= \text{(Selling Price)(Quantity)} \\ R(x) &= px \end{aligned}$$

**Figure 1.4** Graph of a Generic Revenue Function

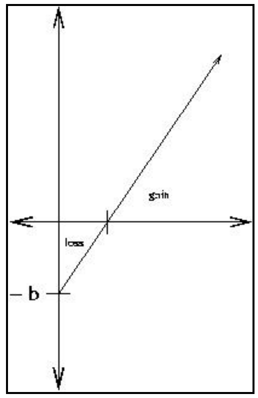


Finally, a **linear profit function** represents the difference between the amount of money a company gains through revenue and the amount of money it pays out in the form of costs.

$$\begin{aligned} P(x) &= R(x) - C(x) = px - (mx + b) = (px - mx) - b \\ &= (p - m)x - b \end{aligned}$$

In this final format, notice that $p - m$ represents the profit for each item made and sold.

**Figure 1.5** Graph of a Generic Profit Function




From Figure 1.5, you can see that when $P(x)=0$, a company is not losing or gaining money. This point is called the **break-even point**.

**Example 1.3** A company is manufacturing and selling insulated mugs. The company has monthly fixed costs of $1500 and there is a total monthly cost of $1800 when producing 100 mugs. Each mug sells for $7.

**a.** Find the cost, revenue, and profit functions for the mug manufacturer, assuming each is a linear function.

**b.** How many mugs must the company make and sell in order to break even?

**Solution**

**a.** First, gather all of the cost information together in ordered pairs of the form $(x, C(x))$. Thus, we have (0, 1500) and (100, 1800). Next, find the slope between the two points:

$$m=\frac{1800-1500}{100-0}=\frac{300}{100}=3$$

Since a linear cost function has the form $C(x)=mx+b$, and we know the values of $m$ and $b$ (b is the fixed cost), the cost function is $C(x)=3x+1500$.

To find the revenue function, all we need is the selling price, which is given as $7. Therefore, the revenue function is $R(x)=7x$.

Finally, the profit function can be found using the cost and revenue functions we just found, as follows:

$$P(x)=R(x)-C(x)=7x-(3x+1500)=4x-1500$$

**b.** The company will break even when $P(x)=0$ (the $x$-intercept of the profit function). Therefore,

$$\begin{aligned} P(x)=4x-1500&=0\\ 4x&=1500\\ x&=\frac{1500}{4}=375 \end{aligned}$$

So, by making and selling 375 mugs, the company will neither gain nor lose money. ◈

**Example 1.4** A company has costs given by $C(x) = 55.5x + 1000$ and revenue given by $R(x) = 80x$, where $x$ represents the number of items the company makes and sells. Find the number of items the company needs to produce in order to break even.

**Solution** A company breaks even when $P(x) = 0$. Therefore, finding profit,

$$P(x) = R(x) - C(x) = 80x - (55.5x + 1000) = 24.5x - 1000$$

and setting it equal to zero we have

$$P(x) = 24.5x - 1000 = 0$$
$$x = \frac{1000}{24.5} \approx 40.82$$

Thus, if it were possible to produce and sell a fraction of an item then they would break even at $^{2000}/_{49}$ items. However, they most likely cannot produce and sell fractional items so this means that the company will never truly break even. (If they sell 40 items they will lose a slight amount of money, but if they sell 41 items they will make a slight amount of money.) ◈

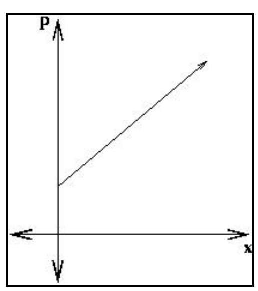
## ECONOMIC APPLICATIONS

Another application of linear functions can be seen in the discussion of supply and demand. A **linear supply function** tells us the price at which a producer is willing to supply exactly $x$ units of a product to the marketplace.

$$S(x) = p = mx + b$$

Generally, because producers are trying to make money, as the price they are given for each product increases, they are willing to increase the amount of product they supply to the marketplace. Thus, as $x$ increases, so does $p$ (or $S(x)$), as shown in Figure 1.6.

**Figure 1.6** Graph of a Generic Supply Function




**Example 1.5** Suppliers are willing to provide 20 light bulbs at a price of 30 cents a piece and are willing to supply 50 light bulbs for 60 cents each. Find the linear supply function describing the light bulb market.

**Solution** Since the supply function is a linear function of the form $S(x) = mx + b$, we can organize the given information as ordered pairs of the form $(x, S(x))$. Thus, we have (20, 0.30) and (50, 0.60) as two points on the line. Now we can find the slope between the two points

$$m = \frac{0.60 - 0.30}{50 - 20} = \frac{-0.30}{30} = \frac{1}{100}$$

and use the point-slope formula of a line to find the supply function as follows:

$$S(x) - 0.60 = \frac{1}{100}(x - 20)$$
$$S(x) = \frac{1}{100}x + \frac{2}{5}.$$

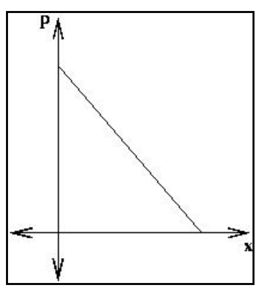
◈

A **linear demand function** tells us the price at which a consumer is willing to buy exactly $x$ units of a product from the marketplace.

$$D(x) = p = mx + b$$

Generally, because consumers are trying to save money, as the price of a product increases, they tend to buy less of that product. Thus, as $x$ increases, $p$ (or $D(x)$) decreases, as shown in Figure 1.7.

**Figure 1.7** Graph of a Generic Demand Function




From Figure 1.6 it is easy to see that the lowest price producers are willing to accept for their product is the $y$-intercept of the linear supply function. Likewise, from Figure 1.7, you can see that the highest price consumers are willing to pay for a product is the $y$-intercept of the linear demand function.

**Example 1.6** Consumers are willing to buy 70 light bulbs at a price of 50 cents a piece and are willing to buy 30 light bulbs for 70 cents each. What is the highest price consumers are willing to pay for a light bulb, assuming a linear demand function?

**Solution** Since the demand function is a linear function of the form $D(x) = mx + b$, we can organize the given information as ordered pairs of the form $(x, D(x))$. Thus, we have (70, 0.50) and (30, 0.70) as two points on the line. Now we can find the slope between the two points

$$m = \frac{0.50 - 0.70}{70 - 30} = \frac{-0.20}{40} = \frac{-1}{200}$$

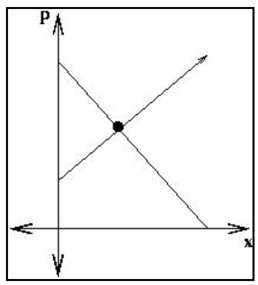
and use the point-slope formula of a line to find the demand function as follows:

$$D(x) - 0.70 = \frac{-1}{200}(x - 30)$$
$$D(x) = \frac{-1}{200}x + \frac{17}{20}.$$

Therefore, the highest price consumers are willing to spend for a light bulb is $\$\,^{17}\!/_{20} = 0.85$. ◈

If you overlay the supply and demand functions, you can find the point where they intersect, called the **equilibrium point**, demonstrating the so called Law of Supply and Demand.

**Figure 1.8** Graph of Market Equilibrium




**Example 1.7** Using the supply and demand functions found for the light bulb market in the previous two examples, find the equilibrium point for the light bulb market.

**Solution** The equilibrium point is found by finding the intersection point of the supply and demand functions of the market. We found the supply function to be $S(x) = 0.01x + 0.1$ and the demand function to be $D(x) = -0.005x + 0.85$. By setting these equal to each other, we can find the equilibrium quantity, *x*, as follows:

$$-0.005x + 0.85 = 0.01x + 0.1$$
$$0.75 = 0.015x$$
$$50 = x.$$

By plugging this value into either the supply or demand function, we can then find the equilibrium price.

$$S(x) = 0.01(50) + 0.1 = 0.60 = p$$

Thus, the equilibrium point for the light bulb market is (50, 0.60), meaning that both consumers and producers will be happy with buying and selling 50 light bulbs at a price of 60 cents a piece. ◈

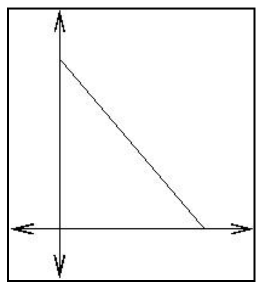
## INVESTMENT APPLICATION

A final application of linear functions can be seen when talking about the depreciation of an object over time. An object depreciates if it loses value over time. If the object loses value at a constant rate, then it is said to be depreciating linearly. A **linear depreciation function** is given by

$$V(t) = mt + b$$

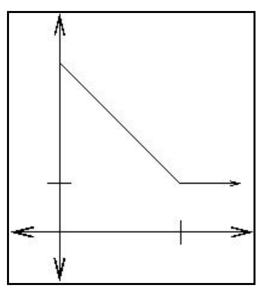
where $t$ is the amount of time over which the object is losing value, $m$ is the rate of depreciation (value lost per time period), and $b$ is the initial value of the object. Notice that $m$ should always be negative since the value is decreasing over time.

**Figure 1.9** Graph of the Value of an Object Over Time



The lowest possible value for an object is $0. However, it is possible for an object to retain some amount of value, indefinitely. The **scrap value** of an object is the lowest value an object obtains.

**Figure 1.10** Non-Zero Scrap Value of an Object



**Example 1.8** Patti buys a car for $17,575. The value of her car linearly depreciates to $5400 over 10 years. Find the value of Patti's car as a linear function of the number of years since she bought the car.

**Solution** The value of the car is a linear function of the form $V(t) = mt + b$, where $b$ represents the value of the car when $t = 0$ (the original value of the car). Therefore, the value of Patti's car is of the form $V(t) = mt + 17575$. To find the value of $m$, use the value of the car when $t = 10$:

$$\begin{aligned} V(10) = m(10) + 17575 &= 5400 \\ 10m &= -12175 \\ m &= -1217.5 \end{aligned}$$

Thus, the value of Patti's car is given by $V(t) = -1217.5t + 17575$ when $0 \le t \le 10$. ◈

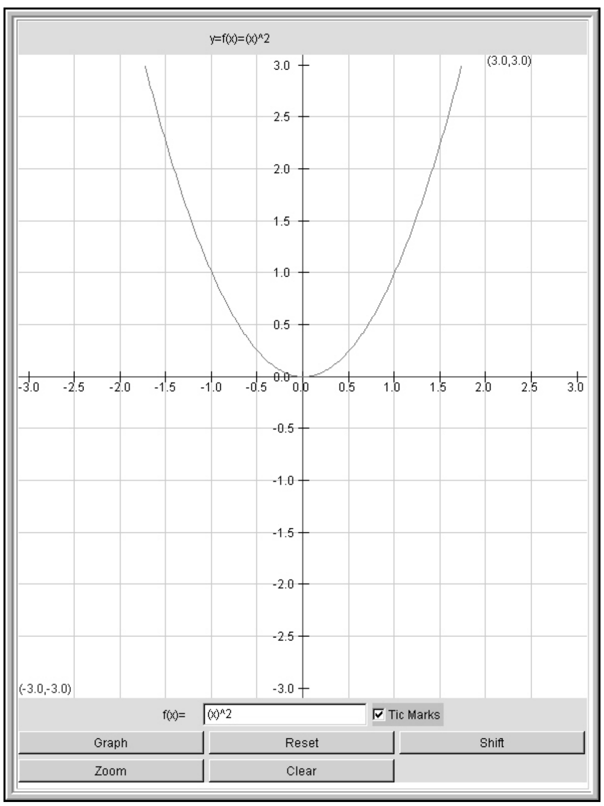
## 1.2 Quadratic Functions

While linear data always increases or always decreases at a constant rate, not all data behaves in such a manner. To account for more complicated situations, we will begin to introduce more complex functions. The first of these functions will be quadratic functions.

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$.

The simplest quadratic function is $f(x) = x^2$ ($a = 1, b = 0, c = 0$). It's graph can be found by using the Plotting Applet and is shown in Figure 1.11.

**Figure 1.11** Graph of $f(x) = x^2$ using the Plotting Applet




The graph of $f(x) = x^2$, and every quadratic function, is known as a **parabola**. Every parabola has a lowest (or highest) point which is known as the **vertex** of the parabola. From Figure 1.11 you can see that the vertex of the graph of $f(x) = x^2$ is at (0, 0). You can also see from Figure 1.11 that if the graph was folded along a vertical line through the vertex $(x = 0)$, the two halves of the parabola would lie exactly on top of one another. This demonstrates the property that all quadratic functions are **symmetric** with respect to the vertical line through the vertex, which is known as the **axis** of the parabola.

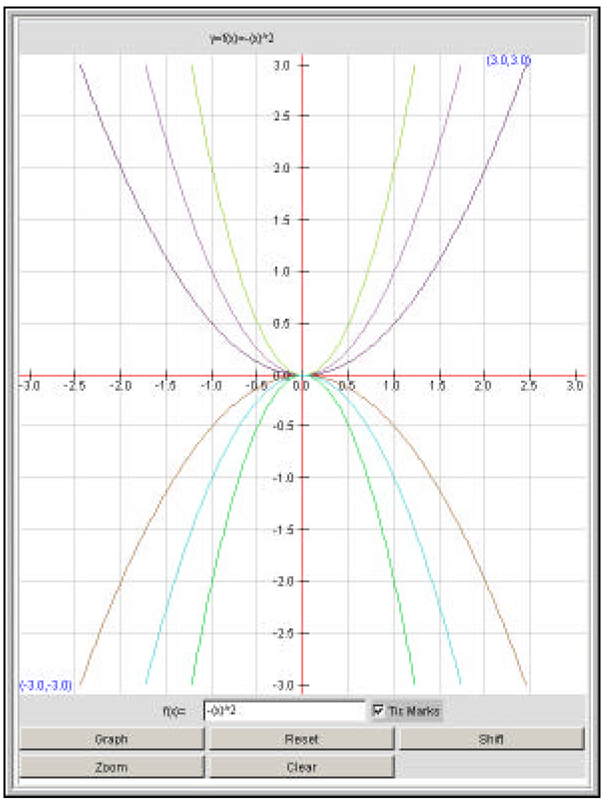
**Example 1.9** Graph the following quadratic functions on the same axes. Studying how does the value a effect the value of a quadratic function.

**a.** $f(x)=x^2$ **c.** $f(x)=2x^2$ **e.** $f(x)=-2x^2$

**b.** $f(x)=\frac{1}{2}x^2$ **d.** $f(x)=-\frac{1}{2}x^2$ **f.** $f(x)=-x^2$

**Solution** Using the Plotting Applet we can graph all six functions, shown in Figure 1.12.

**Figure 1.12** Graphs for Example 1.9 using the Plotting Applet




Notice that the vertex of each of these parabolas is at (0, 0) and the axis of symmetry of each parabola is $x=0$. ◈

Example 1.9 shows the effect of *only* changing the value of *a* in $ax^2+bx+c$. Notice that when $a>0$, the parabola opens upward and the vertex is the lowest point on the graph, while when $a<0$, the parabola opens downward and is the vertex is the highest point on the graph. By comparing the graph of F to A, D to B, and E to C, we can see that in multiplying $f(x)$ by $-1$, the graph of $f(x)$ is reflected over the $x$-axis, which is known as a **vertical reflection** of $f(x)$. By comparing the graphs of B and C to A and the graphs of E and D to F, we can see that the magnitude of $a$ determines how fast the graph of $f(x)$ is increasing or decreasing. When $0<|a|<1$, we say the graph of $f(x)$ is **vertically compressed** by a factor of $a$ and when $|a|>1$, we say the graph of $f(x)$ is **vertically expanded** by a factor of $a$. Finally, when $|a|=1$, we have our simplest parabola, or its reflection across the $x$-axis, which all other parabolas are compared to. These results are summarized in Table 1.1.

**EXAMPLE 1.10** Without graphing, how will the graph of $g(x)=-3x^2$ be different from the graph of $f(x)=x^2$?

**Solution** The only difference between the two functions is the value of $a$. Since $a=-3$, the graph of $g(x)$ can be found by vertically reflecting $f(x)$ and then vertically expanding it by a factor of 3. ◈

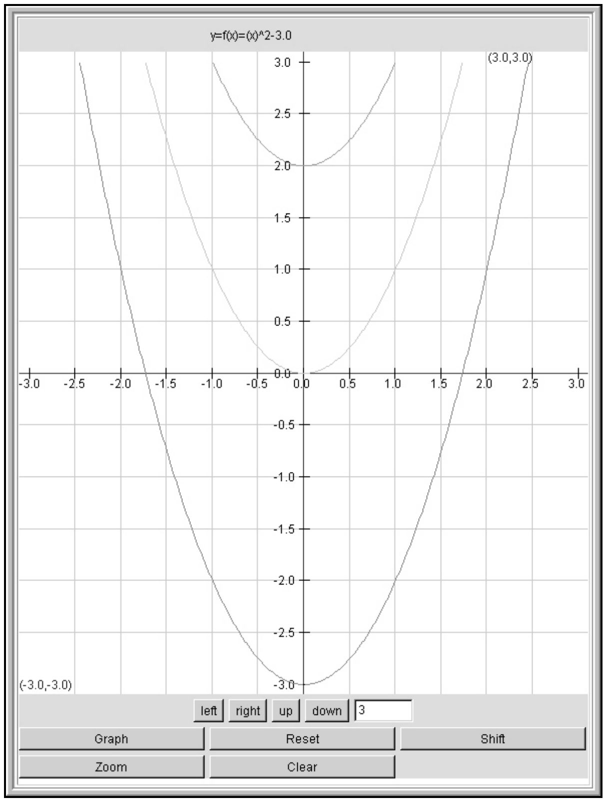
**EXAMPLE 1.11** Graph the following quadratic functions on the same axes and find the vertex of each.

**a.** $f(x)=x^2$ **b.** $g(x)=x^2+2$ **c.** $h(x)=x^2-3$

**Solution** Using the Plotting Applet, we can graph all three functions as shown in Figure 1.13. By examination we see that

**a.** The vertex of $f(x)$ is at (0, 0).
**b.** The vertex of $g(x)$ is at (0, 2).
**c.** The vertex of $h(x)$ is at (0, −3).

**Figure 1.13** Graphs for Example 1.11 using the Plotting Applet




◈

Example 1.11 shows the effect of *only* changing the value of $c$ in $ax^2+bx+c$. If $c$ is increased then the parabola will move up and if $c$ is decreased then the parabola will move down. This type of movement is known as a **vertical shift**. These properties are summarized in Table 1.1.

**EXAMPLE 1.12** Without graphing, how will the graph of $g(x)=x^2+2x+5$ be different from the graph of $f(x)=x^2+2x+1$?

**Solution** The only difference between the two functions is the value of $c$. Since the value of $c$ is increased by 4, the graph of $g(x)$ can be found by vertically shifting the graph of $f(x)$ up 4 units.
In order to compare two quadratic functions where the values of $a$, $b$, and/or $c$ change, it is easier to look at the **standard form** of a quadratic function. By completing the square we get

$$\begin{aligned} f(x) &= ax^2+bx+c = a(x^2+\tfrac{b}{a}x)+c = a(x^2+\tfrac{b}{a}x+(\tfrac{b}{2a})^2)+c-a(\tfrac{b}{2a})^2 \\ &= a(x+\tfrac{b}{2a})^2+(c-\tfrac{b^2}{4a}) = a(x-(\tfrac{-b}{2a}))^2+(c-\tfrac{b^2}{4a}) \\ &= a(x-h)^2+k \end{aligned}$$

where $h=\tfrac{-b}{2a}$ and $k=c-\tfrac{b^2}{4a}=f(h)$. In standard form, the graph of the parabola can be found by shifting the graph of $f(x)=x^2$ horizontally (right if $h>0$, left if $h<0$) by $h$ units, vertically expanding or contracting by a factor of $a$, vertically reflecting if $a<0$, and then shifting vertically by $k$ units. Notice that by performing this transformation, the graph of a quadratic in standard form has a vertex at the point $(h,k)$. ◈

**Table 1.1** Summary Chart for Parabolas

| |
|---|
| Let $f(x) = a(x-h)^2 + k$. Comparing this to the parabola $x^2$, |
| **1.** If $a > 0$ then $f(x)$ opens upward and the vertex is the lowest point. |
| **2.** If $a < 0$ then $f(x)$ vertically reflected and the vertex is the highest point. |
| **3.** If $0 < \lvert a \rvert < 1$ then $f(x)$ is vertically compressed relative to $x^2$. |
| **4.** If $\lvert a \rvert > 1$ then $f(x)$ is vertically expanded relative to $x^2$. |
| **5.** If $k > 0$, then $f(x)$ is vertically shifted upwards by $k$ units. |
| **6.** If $k < 0$, then $f(x)$ is vertically shifted downwards by $k$ units. |
| **7.** If $h > 0$, then $f(x)$ is horizontally shifted to the right by $h$ units. |
| **8.** If $h < 0$, then $f(x)$ is horizontally shifted to the left by $h$ units. |
| **9.** The vertex is at $(h,k)$. |

**Example 1.13** Find the vertex of the parabola $f(x) = 2x^2 - 3x + 6$ and determine whether it is a maximum or minimum.

**Solution** The $x$-coordinate of the vertex is given by

$$x = \frac{-b}{2a} = \frac{-(-3)}{2(2)} = 0.75$$

The $y$-coordinate of the vertex can be found by plugging in the corresponding $x$ value:

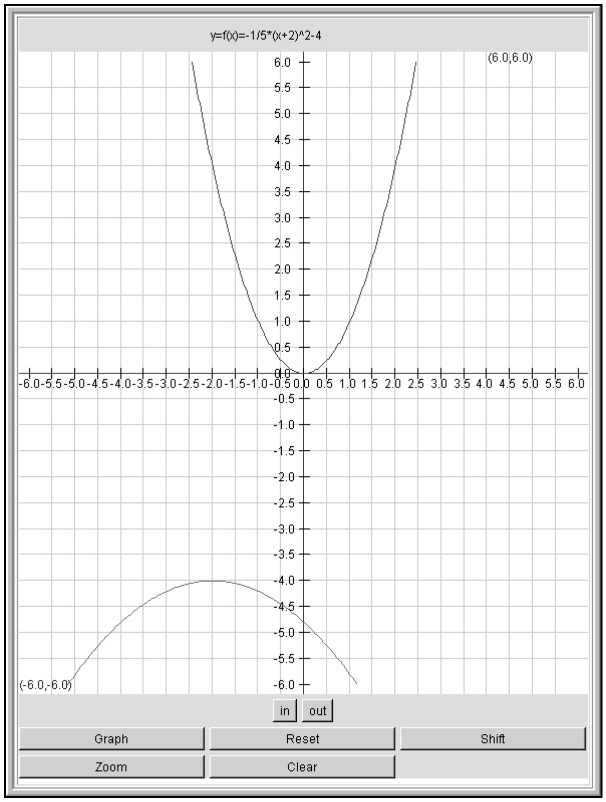
$$f(0.75) = 2(0.75)^2 - 3(0.75) + 6 = 4.875$$

Therefore, the vertex of the parabola is located at (0.75, 4.875). Since $a = 2 > 0$, then the parabola opens upward from the vertex, so the vertex is a minimum. ◈

**Example 1.14** How is the graph of $f(x) = \frac{-1}{5}(x+2)^2 - 4$ different from the graph of $g(x) = x^2$?

**Solution** The graph will be horizontally shifted to the left by 2 units, reflected over the $x$-axis, vertically contracted by a factor of $1/5$, and then vertically shifted down 4 units. Using the Plotting Applet, these transformations can be seen in Figure 1.14.

**Figure 1.13** Graphs for Example 1.11 using the Plotting Applet




◈

Many real-life situations are quadratic in nature, as they portray information that increases to a point and then decreases, such as revenue or profit, or decreases to a point and then increases, such as costs.

**Example 1.15** Given the demand of watches to be $p = -3x + 60$, how many watches must be sold in order to maximize revenue?

**Solution** Recall that revenue is given by $R(x) = px$ where $p$ = price per item sold and $x$ = number of items sold. When price is not fixed, but determined by the buying habits of consumers, price is given as the demand function. Thus,

$$R(x) = px = (-3x + 60)x = -3x^2 + 60x$$

Notice that the revenue function is a quadratic function. Because $a = -3 < 0$, the revenue function will be a parabola that opens downward when it is graphed. Thus, the vertex is the maximum point on the revenue function. The $x$ coordinate of the vertex is given by

$$x = \frac{-b}{2a} = \frac{-60}{2(-3)} = 10$$

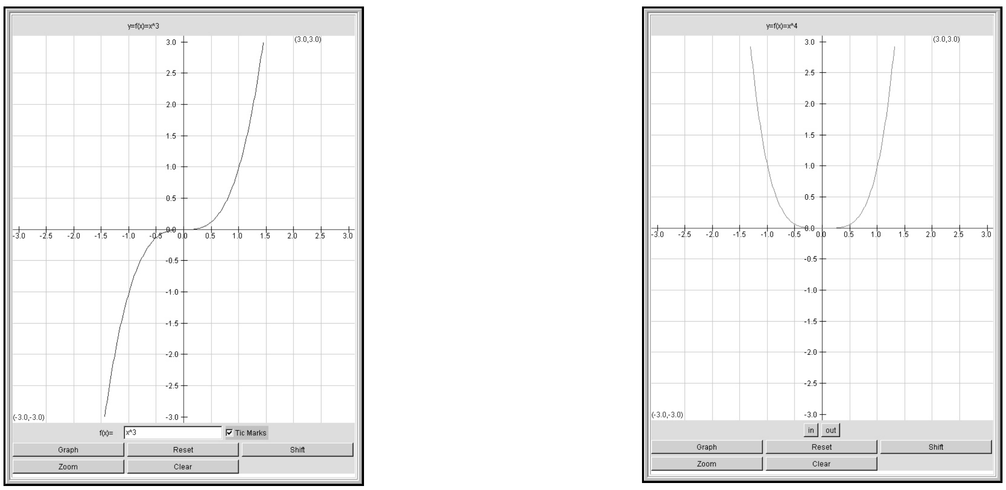
Therefore, 10 watches must be sold in order to maximize revenue. ◈

# 1.3 Cubic and Higher Polynomials

Linear and quadratic functions are two functions belonging to a family of functions called **polynomials**. In general, a polynomial is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_0, a_1, \ldots a_n$ are real numbers and $n$ is a whole number, giving the **degree** of the polynomial. With this definition, linear functions $(f(x) = ax + b)$ are known as first-degree polynomials, while quadratic functions $(f(x) = ax^2 + bx + c)$ are known as second-degree polynomials.

Two higher degree polynomials that will be encountered often are third-degree polynomials (also known as **cubic functions**) and fourth-degree polynomials (also known as **quartic functions**). Their most basic graphs are shown in Figure 1.12.

**Figure 1.15** Graphs of Cubic and Quartic Functions




To determine how polynomial functions behave when their $x$ values become negatively and positively large (as $x$ approaches negative and positive infinity), without looking at their graphs (since they can be hard to graph without the help of technology), it is only necessary to look at the leading term, $a_n x^n$. Tables 1.2 and 1.3 list the long range behavior of polynomials.

**Table 1.2** Summary Charts for Long Range Function Behavior of **Even-Degree** Polynomials

| Leading Coefficient | As $x \to -\infty$ | As $x \to \infty$ |
|---|---|---|
| $a_n > 0$ | $f(x) \to \infty$ | $f(x) \to \infty$ |
| $a_n < 0$ | $f(x) \to -\infty$ | $f(x) \to -\infty$ |

**Table 1.3** Summary Charts for Long Range Function Behavior of **Odd-Degree** Polynomials

| Leading Coefficient | As $x \to -\infty$ | As $x \to \infty$ |
|---|---|---|

| | | |
|---|---|---|
| $a_n > 0$ | $f(x) \to -\infty$ | $f(x) \to \infty$ |
| $a_n < 0$ | $f(x) \to \infty$ | $f(x) \to -\infty$ |

**Example 1.16** Describe the end behavior of the following functions

**a.** $f(x) = 2x^5 - 4x^4 + 3x^2 - 6$

**b.** $g(x) = -3x^4 + 6x - 9$

**c.** $h(x) = -4x^7 - 8x^2 + 2$

**Solution**

**a.** $f(x)$ is an odd-degree polynomial since $n = 5$, and because $a_n = 2 > 0$, we know as $x \to \infty$, $f(x) \to \infty$ and we know as $x \to -\infty$, $f(x) \to -\infty$.

**b.** $g(x)$ is an even-degree polynomial since $n = 4$, and because $a_n = -3 < 0$, we know as $x \to \infty$, $f(x) \to -\infty$ and we know as $x \to -\infty$, $f(x) \to -\infty$.

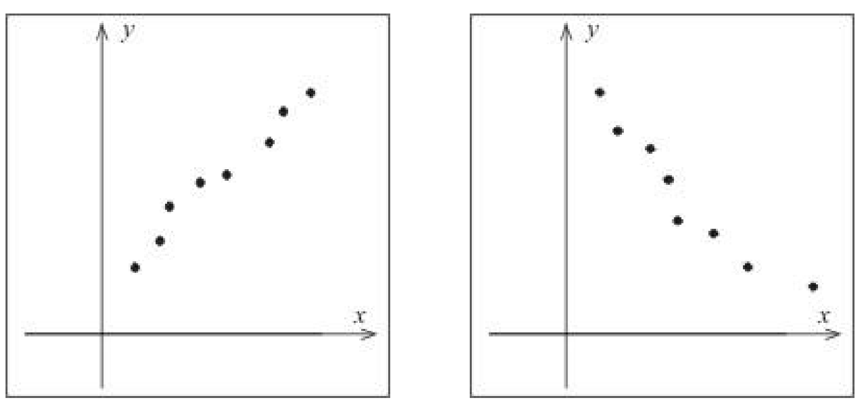
**c.** $h(x)$ is an odd-degree polynomial since $n = 7$, and because $a_n = -4 < 0$, we know as $x \to \infty$, $f(x) \to -\infty$ and we know as $x \to -\infty$, $f(x) \to \infty$. ◈

## 1.4 Modeling with Polynomials

In certain situations, data is collected and used to describe a phenomenon that is occurring. In order to describe the situation accurately and be able to make predictions of future occurrences, a mathematical model is needed. In this section we will discuss the use of polynomial models.

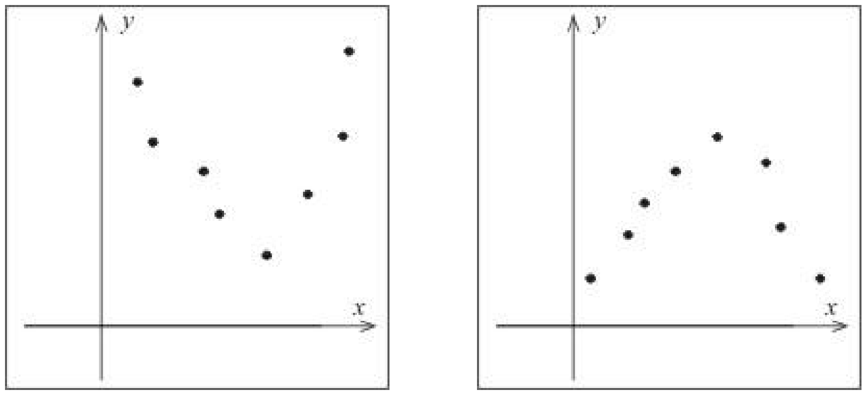
When thinking of using a linear model, the data should fall, more or less, in a straight line and be always increasing or always decreasing. Figure 1.16 shows examples of linear data sets.

**Figure 1.16** Graphs of Linear Data



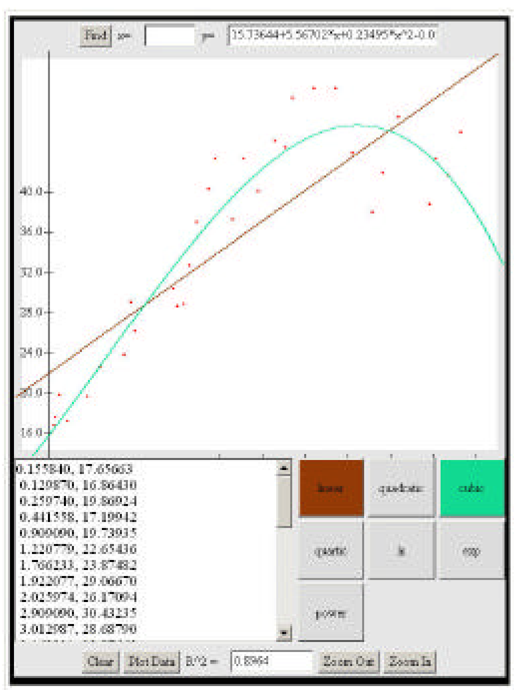
When data increases and then decreases or decreases and then increases, a quadratic model should be explored. Figure 1.17 shows quadratic data.

**Figure 1.17** Graphs of Quadratic Data




If the behavior is more complex, even higher order polynomials can be used to fit the data. In order to find a model that explains a set of given data, you can use the Modeling applet.

**Figure 1.18** Modeling Applet




Notice that in addition to giving you the equation of the model at the top of the applet, you are also given an $R^2$ value at the bottom of the applet. This value is the square of the correlation coefficient. The **correlation coefficient** tells you how well your model explains the data and thus, how accurate it would be at predicting values outside of your data set. The closer, in magnitude, the correlation coefficient is to 1, the better your model is at explaining your data. Thus, an $R^2$ value close to 1 tells you that the model is a good model for your data.

**Example 1.17** The table below represents the percentage of females (16 or older) that were a part of the civilian labor force in January of the respective years.

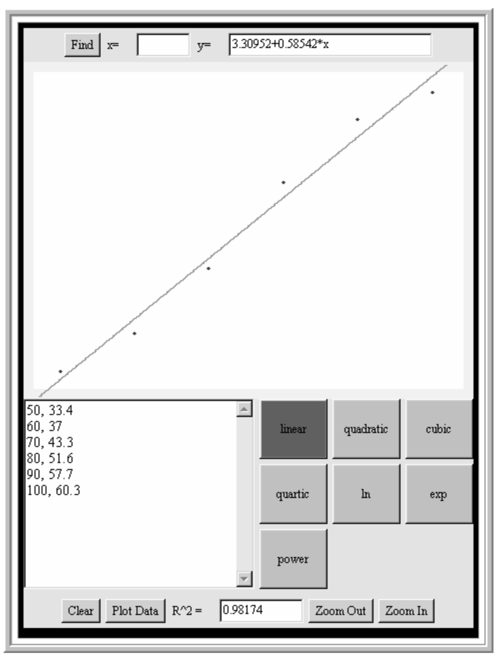
| **Year** | 1950 | 1960 | 1970 | 1980 | 1990 | 2000 |
|---|---|---|---|---|---|---|
| **Percent of Working Females** | 33.4 | 37 | 43.3 | 51.6 | 57.7 | 60.3 |

Source: http://data.bls.gov/servlet/SurveyOutputServlet

**a.** Find the best-fitting linear model to this data.
**b.** How accurate is the model you found?
**c.** Use your model, to predict the percentage of working females in the year 2010, if this employment trend continues.

**Solution**
**a.** We will let the independent variable, $x$, represent the year, where the value of $x$ is given as the number of years since 1900. (i.e. 50 represents the year 1950). The dependent variable, $y$, will represent the percentage of working females. The results of inputting the data into the Modeling Applet and clicking on the linear regression button is shown in Figure 1.19

**Figure 1.19** Linear Fit for Example 1.17

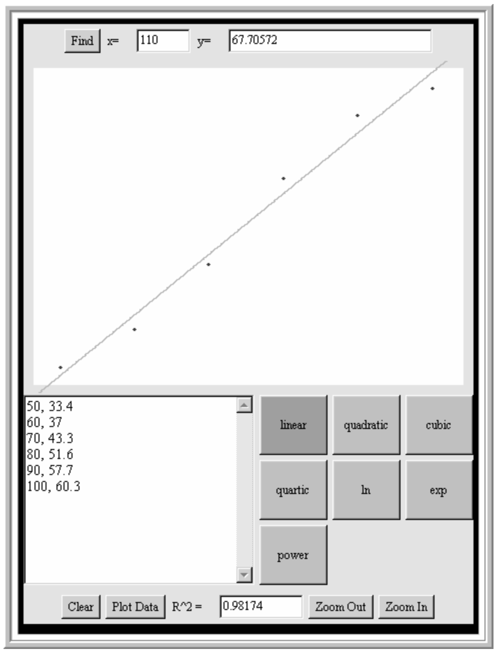



Therefore, $y = 0.58542x + 3.30952$ is the best-fitting linear model to the data.

**b.** The $R^2$ value for this model is given as 0.98174, which is very close to 1, and, therefore, shows this model to be an accurate model for the data.

**c.** What will be happening in the year 2010 is equivalent to finding the *y*-value corresponding to $x = 110$. Using the Modeling Applet we get

**Figure 1.20** Modeling Applet with $x = 110$




so, about 67.7% of females will be a part of the civilian labor force, if this trend continues. ◈

When you are not told which model to find (as is the case in real life), you must decide which model best represents the data you are given. Moreover, you are not limited to only linear models. We will restrict our discussion in this section to polynomial models, but you will encounter the use of other models in later chapters, as different functions are introduced.

When trying to determine which model best represents your data, first look at a scatterplot of your data and determine the possible shape your model should have. (Remember to think of the end behavior your model should have for predicting values outside of your given data set.) Second, find all models which fit the shape of your data and plot each model with your data. (Remember to zoom out to see the end behavior of your models.) Last, compare the models to find the best fit. You are looking for the simplest model which accurately predicts the given situation.

**Example 1.18** The following table gives the 30-year fixed-rate conventional mortgage interest rates from 1972 to 1982.

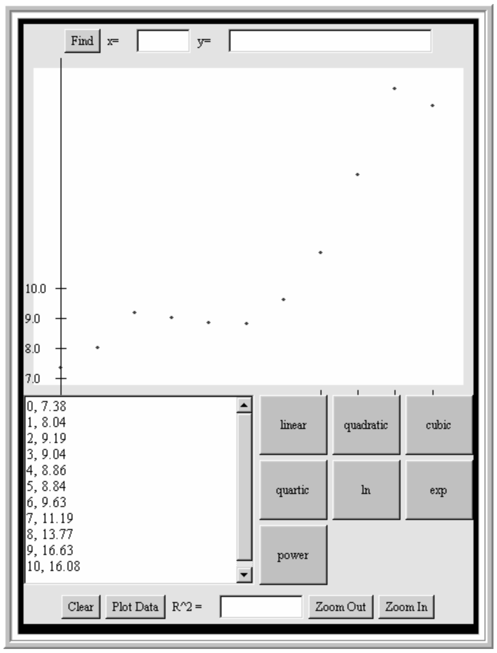
| **Year** | 1972 | 1973 | 1974 | 1975 | 1976 | 1977 | 1978 | 1979 | 1980 | 1981 | 1982 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Interest Rate** | 7.38 | 8.04 | 9.19 | 9.04 | 8.86 | 8.84 | 9.63 | 11.19 | 13.77 | 16.63 | 16.08 |

Source: http://www.federalreserve.gov/releases/h15/data/a/cm.txt

Find the best-fitting polynomial model to this data and explain why the model you chose is the best model.

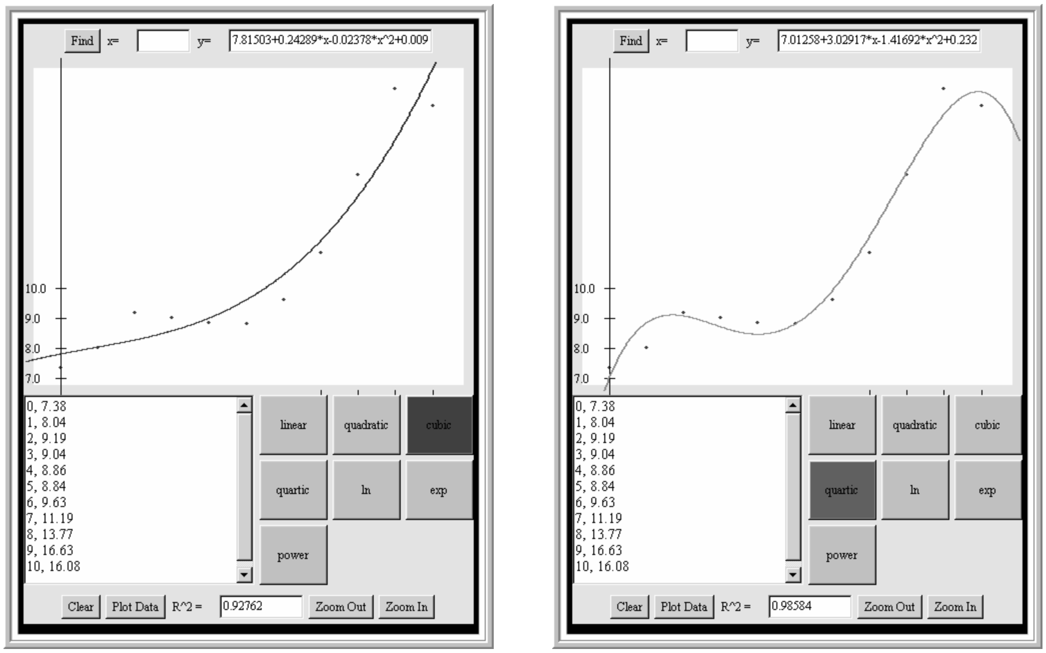
✔ **Solution** We will let the independent variable, $x$, represent the year, where the value of $x$ is given as the number of years since 1972. (i.e. 0 represents the year 1972). The dependent variable, $y$, will represent the interest rate. We will first input the data into the Modeling Applet and look at the scatterplot of the data. The results are shown in Figure 1.21.

❏ **Figure 1.210** Interest Rate Data




Since the data does not strictly increase, we can rule out using a linear model. Likewise, it begins by increasing and then decreases, but then increases again so a quadratic model would not seem to be the best fit for this data. The only two polynomial models worth trying are cubic and quartic shown below.

**Figure 1.22** Modeling Applet with Cubic and Quadratic Fits.




It is clear that even though quartic is the most complicated polynomial model, it best represents the given data set. Thus, the best-fitting model is

$$y = -0.01114x^4 + 0.23231x^3 - 1.41692x^2 + 3.02917x + 7.01258$$ ◈

# Sample Quiz

**Question 1.1** Given $f(x) = -\frac{2}{3}x + 5$, if $x$ decreases by 9, what is the corresponding change in the function value?

**Question 1.2** Small fish bowls sell for $5.00. The company making the fish bowls has total costs of $550 when making 50 fish bowls and $650 when making 100 fish bowls. Find the profit function of the company making the fish bowls.

**Question 1.3** At a price of $40 per book, 500 books can be sold and at a price of $30 per book, 100 books can be sold.

**a.** Assuming linear demand, find the demand equation as a function of the number of books sold.

**b.** What is the highest price consumers are willing to pay for this book?

**Question 1.4** Kathryn bought a brand new car in 1999 for $20,500. In 2002 it is only worth $13,285. Assuming linear depreciation is occurring, what will her car be worth in 2006?

**Question 1.5** Find the vertex of $f(x) = 3x^2 - 6x + 7$.

**Question 1.6** How will the graph of $g(x) = -3(x-4)^2 + 2$ be different from the graph of $f(x) = x^2$ ?

**Question 1.7** The linear demand function for a particular item is given by $p = -2x + 50$. If the linear cost function for the company producing the item is given by $C(x) = 30x + 40$, find the number of items the company must make and sell in order to maximize it's profits.

**Question 1.8** Describe the end behavior of the function $f(x) = 6x^{10} - 3x^8 + x^7 - 2x + 1$.

**Question 1.9** The following table gives the life expectancy of females at birth (in years) for the given years.

| **Year** | 1929 | 1939 | 1949 | 1959 | 1969 | 1979 | 1989 | 1999 |
|---|---|---|---|---|---|---|---|---|
| **Life Expectancy** | 58.7 | 65.4 | 70.7 | 73.2 | 74.4 | 77.8 | 78.5 | 79.4 |

Source: National Vital Statistics Report, Vol. 50, No. 6, March 21,2002

Find the equation of the best-fitting quadratic model to this data.

**Question 1.10** The following table gives the median income (in dollars) for a four-person family living in Texas between 1990 and 2000.

| **Year** | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Income** | 37,789 | 39,204 | 40,342 | 40,688 | 42,570 | 43,977 | 46,757 | 48,007 | 51,148 | 53,291 | 53,513 |

Source:

Which polynomial model (linear, quadratic, cubic, or quartic) best fits the given data and why?

# Chapter 2 Exponentials and Logarithms

The exponential function is one of the most important functions in the field of mathematics. It is widely used in a variety of applications such as compounded interest, population growth, and carbon dating. This chapter is a brief introduction to the exponential function and its inverse function, the logarithmic function. We will look at various properties of these two functions, learn to solve equations involving them, and then finally use them to model real world situations.

## 2.1 Exponential Functions

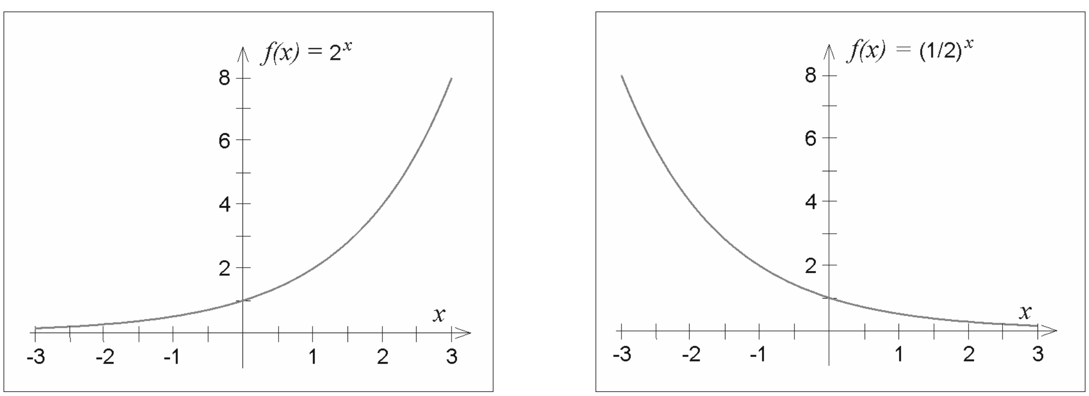
An exponential function is a function of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$. Some properties of exponential functions are summarized in Table 2.1. Figure 2.1 shows the two different types of curves that are generated by an exponential function.

**Table 2.1** Properties of Exponential Functions

Let $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

1. If $b > 1$ then $f(x)$ is an exponential growth function.
2. If $0 < b < 1$ then $f(x)$ is an exponential decay function.
3. The domain of $f(x)$ is $(-\infty, \infty)$.
4. The range of $f(x)$ is $(0, \infty)$.
5. The $y$-intercept is (0, 1).

**Figure 2.1** Graphs of exponential growth and decay functions.



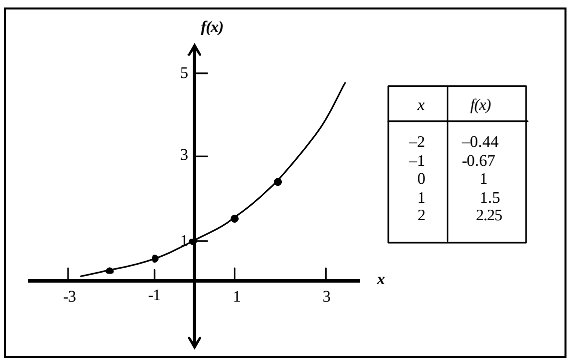
**EXAMPLE 2.1** Determine whether each of the following are exponential growth or decay functions. Then graph each function and verify the domain, range and $y$-intercept of the function are as stated in Table 2.1.

**a.** $f(x)=\left(\frac{3}{2}\right)^x$ **b.** $f(x)=\left(\frac{1}{4}\right)^x$

**Solution**

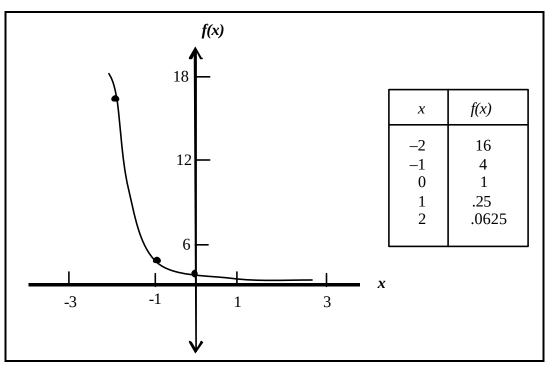
a. This is an exponential growth function because $b=1.5$ and that is greater than 1.

**Figure 2.2** Graph of Example 2.1 **a**. By inspection of the graph in Figure 2.2, the domain is $(-\infty,\infty)$ and the range is $(0,\infty)$.




**b.** This is an exponential decay function because $b=0.25$ and that is less than 1.

**Figure 2.3** Graph of Example 2.1 **b**. By inspection of the graph in Figure 2.3, the domain is $(-\infty,\infty)$ and the range is $(0,\infty)$.




◈

The function $f(x)=2^x$ is an exponential function because the independent variable, $x$, is part of the exponent. If we were given a table of values, or a data set, how could we determine if it modeled an exponential function? One method is to calculate the ratio of successive $f(x)$ values when the $x$ values are equally spaced. If the ratios are all the same (or close to the same) the table of values would model an exponential function. The value of the successive ratios is the base of the exponential function.

**EXAMPLE 2.2** Determine if the table of values given below represents an exponential function. If so, find the base of the function.

| | | | | | |
|---|---|---|---|---|---|
| *x* | 2 | 3 | 4 | 5 | 6 |
| *f(x)* | 6.25 | 15.625 | 39.063 | 97.656 | 244.14 |

**Solution** Compute the successive ratios, as shown below.

$$\frac{f(3)}{f(2)}=\frac{15.625}{6.25}=2.5,\quad \frac{f(4)}{f(3)}=\frac{39.063}{15.625}\approx 2.5,\quad \frac{f(5)}{f(4)}=\frac{97.656}{39.063}\approx 2.5,\quad \frac{f(6)}{f(5)}=\frac{244.14}{97.656}=2.5$$

Since all ratios are approximately equal to 2.5 we conclude that the table of values does represent an exponential growth function with a base of 2.5. ◈

Exponential functions have a variety of applications in the business world and it is often necessary to solve these equations for different variables. In order to solve these equations we must know the laws and definitions associated with exponential functions that are listed in Table 2.2.

**Table 2.2** Properties and Laws of exponents ***a,b 0, positive??

Let *a* and *b* be positive numbers and let *m* and *n* be real numbers. Then,

**1.** $a^0 = 1$

**2.** $a^{-m} = \frac{1}{a^m}$, where $a \neq 0$

**3.** $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

**4.** $a^m \cdot a^n = a^{m+n}$

**5.** $\frac{a^m}{b^n} = a^{m-n}$

**6.** $(ab)^m = a^m b^m$

**7.** $(a^m)^n = a^{m \cdot n}$

**8.** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

**Example 2.3** Use the laws and definitions of exponents from Table 2.2 to solve the following equations for *x*.

**a.** $5^{x+3} = 5^{2x}$

**b.** $8^{x} = 64^{2x-5}$

**c.** $3^{x+1} - \left(\frac{1}{27}\right)^{2x} = 0$

**d.** $x \cdot 9^{-x} = \left(\frac{x}{3^{x}}\right)^{2}$

**e.** $9^{x} - 12 \cdot 3^{x} = -27$

✔ **Solution**

**a.** The expressions on both sides of the equation have a base of 5. Since these expressions and the bases are equal we can conclude that their exponents must also be equal. So we set up an equation stating this fact and solve it for *x*.

$$\begin{aligned} x + 3 &= 2x \\ 3 &= x \end{aligned}$$

**b.** Here the bases of each term are not equal to each other. Thus, our first step is to search for a common base. We know that $64 = 8^{2}$ so we can rewrite 64 as $8^{2}$. Use the laws from Table 2.2 to find

$$8^{x} = \left(8^{2}\right)^{2x-5} = 8^{2(2x-5)} = 8^{4x-10}$$

Now solve for *x* by setting the exponents equal,

$$\begin{aligned} x &= 4x - 10 \\ 10 &= 3x \\ \frac{10}{3} &= x \end{aligned}$$

c. Since the bases on each term are different we must rewrite 27 as $3^{3}$ and apply laws and definitions of exponents,

$$3^{x+1} - \left(\frac{1}{3^{3}}\right)^{2x} = 3^{x+1} - \left(3^{-3}\right)^{2x} = 3^{x+1} - 3^{-6x} = 0$$

Now solve for *x*,

$$\begin{aligned} 3^{x+1} &= 3^{-6x} \\ x + 1 &= -6x \\ 7x &= -1 \\ x &= -\frac{1}{7}. \end{aligned}$$

**d.** This problem differs from the previous three because there is an *x* in each term that accompanies the exponential function. Our strategy in solving this problem will still involve getting the same bases, but now we must factor the expression and solve using zero product property[1]. Begin with the left side of the equation and writing 9 as $3^{2}$,

[1] That is, in a series of products, if any one of the terms are zero, then the entire product is zero.

$$x \cdot 9^{-x} = x\left(3^2\right)^{-x} = x \cdot 3^{-2x}$$

Now work with the right side of the equation,

$$\left(\frac{x}{3^x}\right)^2 = \frac{x^2}{\left(3^x\right)^2} = \frac{x^2}{3^{2x}} = x^2 3^{-2x}$$

Next set them equal to each other,

$$x3^{-2x} = x^2 3^{-2x}$$

Rearrange to have all the variables on the left and factor.

$$x3^{-2x} - x^2 3^{-2x} = 0$$
$$x3^{-2x}(1-x) = 0$$

Now by applying the zero product property we obtain:

$$x = 0,\ 3^{-2x} = 0, \text{ and } 1 - x = 0.$$

Since an exponential function can never equal zero, $3^{-2x} \neq 0$, the only solutions are $x = 0, 1$.

**e.** Since each exponential function is accompanied by an $x$, we will use the strategy from the previous problem. That is, we will get the bases the same and then factor.

$$\left(3^x\right)^2 - 12 \cdot 3^x = -27$$

Let $u = 3^x$ for ease in factoring and find

$$\left(3^x\right)^2 - 12 \cdot 3^x + 27 = u^2 - 12u + 27 = (u-9)(u-3) = 0$$

The first term has the solution $u = 9$ or $3^x = 9 = 3^2$, so $x = 2$. The second term has the solution $u = 3$ or $3^x = 3 = 3^1$, so $x = 1$. Thus, the two solutions to the given equation are $x = 1, 2$.

A check of all these answers through substitution of the solution value into the original equation is encouraged. ◈

# 2.2 Applications of Exponential Functions

One of the most common types of exponential functions in the banking industry is compound interest, which can be easily derived from the simple interest formula.

**Simple Interest**: If $P$ dollars is deposited into an account that earns an annual interest of $r$% (expressed as a decimal), then the amount of interest accumulated, $I$, after $t$ years is given by

$$I = Prt$$

Suppose you accumulated $1,000 in cash from your high school graduation. If you deposit this money into an account that earns 5% simple interest, then at the end of 4 years you would have earned

$$I = (1{,}000)(0.05)(4) = \$200.$$

So you would have $1,000 + $200 = $1,200 in the account at the end of 4 years. In general, the amount in an account that earns simple interest for t years is

$$\begin{aligned} A &= P + I \\ A &= P + Prt \\ A &= P(1+rt) \end{aligned}$$

Another type of interest commonly used in the banking industry is compound interest. This type of interest is often computed more than once a year and the account earns interest on the interest computed the previous compounding period. This is what makes compound interest more appealing to investors than simple interest. For example, if you take the $1,000 accumulated from your high school graduation and deposit it into an account that earns 5% compounded quarterly, we can compute the amount in the account at the end of 1 year. To solve this we use $A = P(1+rt)$ with $t = 1/4$ because the interest is figured quarterly ( $1/4$ of a year).

| Quarter | $A = P(1 + rt)$ | Amount at the end of the Quarter |
|---|---|---|
| 1st | $A = 1{,}000(1 + 0.05(1/4)) = 1{,}000(1.0125)$ | $1,012.50 |
| 2nd | $A = 1{,}012.5(1.0125) = 1{,}000(1.0125)(1.0125) = 1{,}000(1.0125)^2$ | $1,025.16 |
| 3rd | $A = 1{,}025.16(1.0125) = 1{,}000(1.0125)^2(1.0125) = 1{,}000(1.0125)^3$ | $1,037.97 |
| 4th | $A = 1{,}037.97(1.0125) = 1{,}000(1.0125)^3(1.0125) = 1{,}000(1.0125)^4$ | $1,050.95 |

If we look at the expression for $A$, near the end of the formula we can see a pattern that is developing for compound interest. The exponent on 1.0125 is the same as the compounding period. Thus for the 16th compounding period (4 years) the amount in the account would be $1{,}000(1.0125)^{16} = 1{,}219.89$. If we compare this number to the amount in the account that earned simple interest for 4 years, we see that the compounded account earned $19.89 more. This difference is greater for larger initial deposits and when the money is left in the accounts for longer time periods.

It is without a doubt that the computation above was long and tedious. We would not want to use this method of computing for larger numbers, therefore we generalize the computation and derive a formula for compound interest.

**Compound Interest Formula:** If $P$ dollars is deposited into an account earning $r$% annual interest (expressed as a decimal) and is compounded n times a year, then the amount in the account, $A$, after $t$ years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Interest can be compounded a number of different ways. Below are the $n$ values for the different compounding periods.

**Table 2.3** Compounding Periods

| | Frequency, $n$ |
|---|---|
| Annually | 1 |
| Semi-annually | 2 |
| Quarterly | 4 |
| Monthly | 12 |
| Weekly | 52 |
| Daily | 365 |

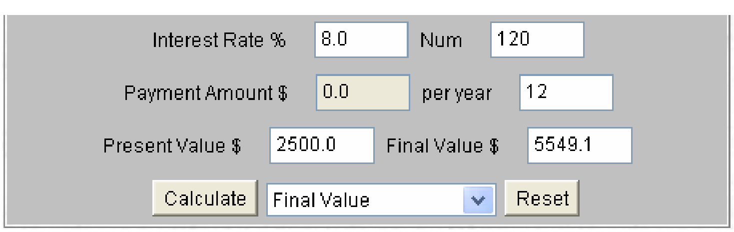
**Example 2.4** Find the amount in an account after 10 years if $2,500 is compounded monthly at 8%.

**Solution** Here $P = 2{,}500,\ t = 10,\ n = 12,\ r = 0.08$. Thus, the amount, A, would be

$$A = 2{,}500\left(1 + \frac{0.08}{12}\right)^{(12)(10)} \approx \$5{,}549.10$$

Or, we can us the compound interest calculator applet to find the same result as shown in Figure 2.4.

**Figure 2.3** Compound Interest Applet for Example 2.4



**Example 2.5** How much money should be deposited into an account that earns 6.5% compounded semi-annually if, after 7 years, the account must be worth $10,000?

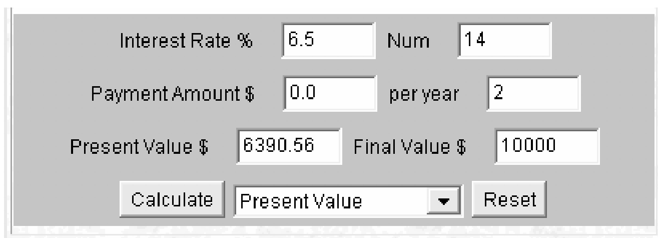
**Solution** Here $A = 10{,}000$, $r = 0.065$, $n = 2$, $t = 7$. Plugging in these values into the formula, we have

$$10{,}000 = P\left(1 + \frac{0.065}{2}\right)^{(2)(7)} = P(1.0325)^{14}.$$

Now solve for P,

$$P = \frac{10{,}000}{(1.0325)^{14}} \approx \$6{,}390.56.$$

**Figure 2.4** Compound Interest Applet for Example 2.5




◈

Suppose you invest $1 in an account that earns 100% interest for 1 year. How would the amount of money at the end of the year change if the interest was compounded more and more often? The compound interest formula with these values would be

$$A = 1\left(1 + \frac{1}{n}\right)^n$$

In Table 2.4 we see how more frequent compounding (larger $n$) gives you a larger and larger $A$, up to a point.

**Table 2.4** Amount of Money vs. Compounding Frequency

| $n$ | 1 | 10 | 100 | 1,000 | 10,000 | 100,000 | 1,000,000 |
|---|---|---|---|---|---|---|---|
| $A$ | 2 | 2.5937 | 2.7048 | 2.7169 | 2.7181 | 2.7182 | 2.7182 |

We see that as $n$ grows larger, $A$ becomes approximately 2.7182. This number (called $e$) is frequently used in the field of mathematics and is used in the formula for continuously compounded interest.

**Continuously Compounded Interest**: If $P$ dollars is deposited into an account earning $r$% annual interest (expressed as a decimal) and is compounded continuously, then the amount in the account, $A$, after $t$ years is

$$A = Pe^{rt}$$

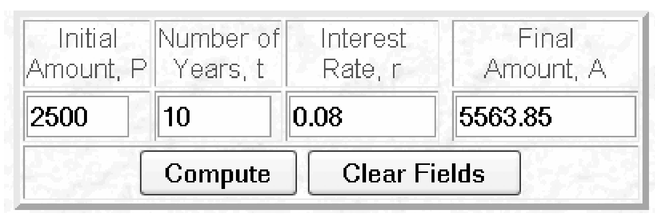
**Example 2.6** Find the amount in an account after 10 years if $2,500 is deposited into an account that earns 8% compounded continuously.

**Solution** Here $P$ = 2,500, $r$ = 0.08, and $t$ = 10. When we substitute into the continuous compound interest formula we get:

$$A = 2{,}500e^{(0.08)(10)} \approx 5{,}563.85$$

In the module there is a continuous compounding calculator that will give the same result, shown below,

**Figure 2.5** Continuous Compounding Applet for Example 2.6




So there will be approximately $5,563.85 in the account after 10 years. Comparing this answer to the one obtained in Example 2.4 shows that continuously compounded interest earns $14.75 more than compounding quarterly. ◈

# 2.3 Logarithmic Functions

We can solve exponential equations when the bases are the same, but how would we solve an equation like $2^x = 10$? To solve this we need to use the inverse of an exponential function, the logarithmic function.

**Definition of Logarithm**: For all $b > 0$, $b \neq 1$, and $y > 0$, $y = b^x$ if and only if $\log_b y = x$

Note: $\log_b y = x$ is read "log to the base b of y equals x".

Two commonly used logarithmic functions are the common logarithm ($\log_{10}$) and the natural logarithm ($\log_e$). Typically we write log for $\log_{10}$ and ln for $\log_e$.

**Example 2.7** Rewrite $6^2 = 36$ and $e^k = 4$ as logarithmic equations.

**Solution** $6^2 = 36$ is in the exponential form, $y = b^x$ , with $y = 36$, $b = 6$ and $x = 2$. Match this to the logarithmic form, $\log_b y = x$, to find $\log_6 36 = 2$. Matching terms for the exponential form $e^k = 4$ gives $y = 4$, $b = e$ and $x = k$. Put these values into the logarithmic form gives $\log_e 4 = \ln 4 = k$. ◈

**Example 2.8** Rewrite $\log_b 9 = 2$ and $\ln k = 2$ as exponential equations.

**Solution** $\log_b 9 = 2$ is in the logarithmic form, $\log_b y = x$, with $b = b$, $y = 9$ and $x = 2$. Match this to the exponential form and find $b^2 = 9$. Rewrite $\ln k = 2$ as $\log_e k = 2$ and find that $b = e$, $y = k$ and $x = 2$. Put this into the exponential form and find $e^2 = k$. ◈

Figure 2.6 below shows the graph of $y = \log_2 x$ and its corresponding table of values. Notice that the table of values is constructed by using the exponential form of $y = \log_2 x$.

**Figure 2.6** Graph of $y = \log_2 x$.

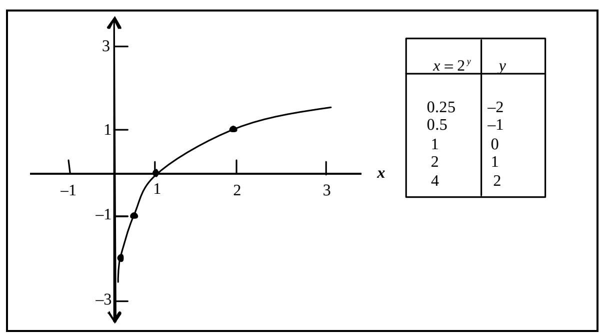



**Table 2.5** Logarithmic Functions

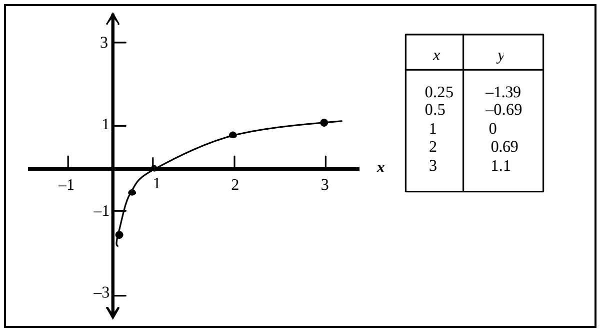
Let $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.
Then,

1. The domain of $f(x)$ is $(0, \infty)$.
2. The range of $f(x)$ is $(-\infty, \infty)$.
3. The $x$-intercept is (1, 0).

**Example 2.9** Graph $f(x) = \ln x$ and verify its domain and range.

**Solution** The graph is shown in Figure 2.7. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

**Figure 2.7** Graph of $y = \ln x$.




To solve equations involving logarithms, we will need to know the Laws of Logarithms listed in Table 2.6.

**Table 2.6** Properties of Logarithms

If $M$, $N$, and $b$ are all positive numbers where $b \neq 1$ and $c$ is any real number, then:

| | |
|---|---|
| **1.** $\log_b MN = \log_b M + \log_b N$ | **5.** $\log_b 1 = 0$ |
| **2.** $\log_b \dfrac{M}{N} = \log_b M - \log_b N$ | **6.** $\log_b b^c = c$ |
| **3.** $\log_b M^c = c \log_b M$ | **7.** $b^{\log_b M} = M$ |
| **4.** $\log_b b = 1$ | |

**Example 2.10** Given $\log_b 4 = 0.7124$ and $\log_b 3 = 0.5645$, use the definition and laws of logarithms to evaluate the following expressions:

**a.** $\log_b 12$ **b.** $\log_b 9$ **c.** $\log_b \dfrac{16}{3}$

**Solution**

**a.** $\log_b 12 = \log_b (4 \cdot 3) = \log_b 4 + \log_b 3 = 0.7124 + 0.5645 = 1.2769$

**b.** $\log_b 9 = \log_b 3^2 = 2\log_b 3 = 2(0.5645) = 1.129$

**c.** $\log_b \left(\dfrac{16}{3}\right) = \log_b \left(\dfrac{4^2}{3}\right) = \log_b 4^2 - \log_b 3 = 2\log_b 4 - \log_b 3 = 2(0.7124) - 0.5645 = 0.8603$ ◈

**Example 2.11** Use the definition and laws of logarithms to solve the following equations for $x$.

**a.** $\ln(x+2) = 4$ **b.** $\log(\log 2x) = 1$

**Solution**

**a.** First rewrite the logarithmic equation as an exponential equation then solve for *x*:

$$e^4 = x+2$$
$$e^4 - 2 = x.$$

Substitute this into the original equation to check:

$$\ln(x+2) = \ln(e^4 - 2 + 2) = \ln(e^4) = 4\ln e = 4\,.$$

**b.** Since this is an embedded logarithm we must rewrite as an exponential equation twice, and then solve for *x*:

$$10^1 = \log 2x$$
$$10^{10} = 2x$$
$$\frac{10^{10}}{2} = x.$$

Check this solution in the original equation,

$$\log\left(\log\left[2\cdot\frac{10^{10}}{2}\right]\right) = \log\left(\log\left[10^{10}\right]\right) = \log\left(10\log\left[10\right]\right) = \log\left(10\cdot 1\right) = 1$$

◈

**Example 2.12** Solve the following exponential equations for *x*.

**a.** $2\left(e^{2x-5}\right) = 4$ **b.** $e^{\sqrt{x}} = 9$ **c.** $10^{\log(2x+1)} = \ln 2$

**Solution**

**a.** Divide both sides of the equation by 2,

$$e^{2x-5} = 2$$

Since the variable is in the exponent, rewrite as a logarithmic equation and then solve for *x*:

$$2x - 5 = \ln 2$$
$$2x = \ln 2 + 5$$
$$x = \frac{\ln 2 + 5}{2} \approx 2.8466.$$

**b.** Rewrite as a logarithmic equation,

$$\ln 9 = \sqrt{x}$$

Solve for *x* by squaring both sides,

$$x = (\ln 9)^2 \approx 4.8278.$$

**c.** Use logarithmic properties from Table 2.6 on the left side of the equation and then solve for *x*,

$$2x + 1 = \ln 2$$
$$2x = \ln 2 - 1$$
$$x = \frac{\ln 2 - 1}{2} \approx -0.1534$$

◈

# 2.4 Modeling

As we have seen in earlier sections, we can model real world data with exponential and logarithmic functions.

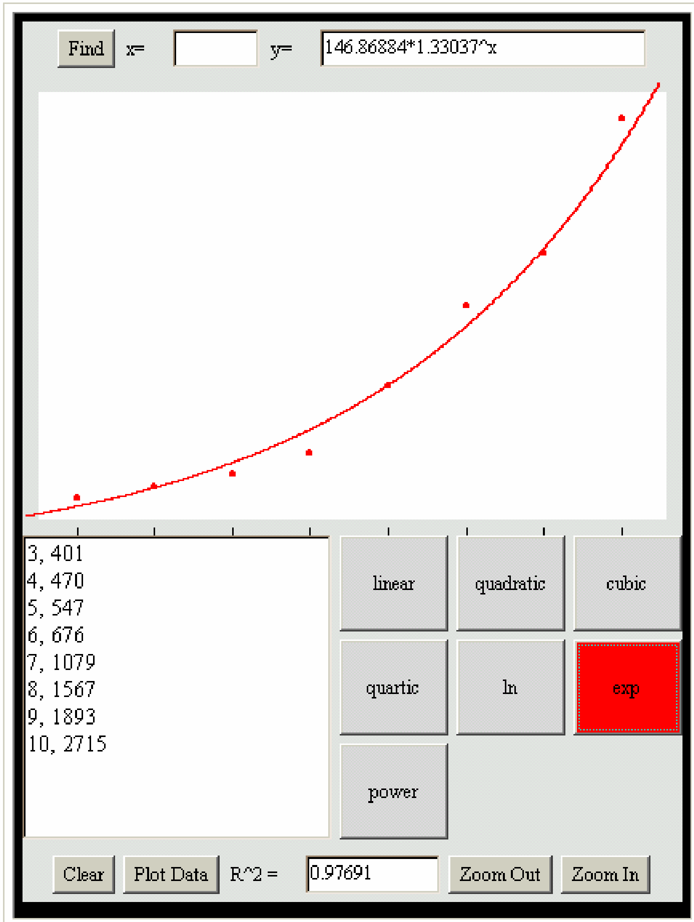
**Example 2.13** The data below represents the amount of goods (in millions of dollars) the United States imported from Hungary from 1993 to 2000. Find the best fitting exponential model for this data set and use the model to predict the amount of imported goods in the year 2010.

| Year | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 |
|---|---|---|---|---|---|---|---|---|
| Amount of goods imported (in millions of dollars) | 401 | 470 | 547 | 676 | 1079 | 1567 | 1893 | 2715 |

Source: http://www.ita.doc.gov/td/industry/otea/usfth/aggregate/HL00T11.html

**Solution** We will make the independent variable, *x*, the import year where the value of *x* represents the number of years after 1990 (i.e. 4 represents 1994). The amount of goods imported will be the dependent variable, *y*. Using the regression calculator available for these modules we input the data, click on plot data, and then choose the exponential regression button. In doing so we obtain $y = 146.87(1.33)^x$ with $R^2 = 0.97691$ as shown in Figure 2.8.

**Figure 2.8** Modeling Applet for Example 2.13




To predict the amount of imported goods in the year 2010 we input $x = 20$ into the regression calculator and find $y = 21{,}474.84$ million dollars. ◈

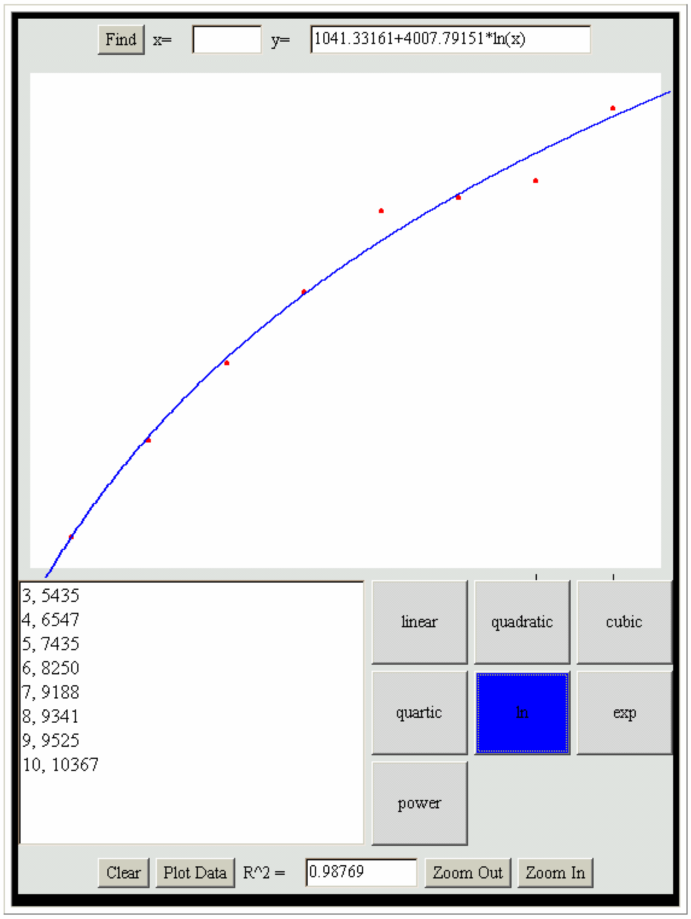
**Example 2.14** The table below represents the amount of goods (in millions of dollars) the United States imported from Indonesia from 1993 to 2000. Find the best fitting logarithmic model for this data set and use the model to predict the amount of imported goods in the year 2005.

| Year | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 |
|---|---|---|---|---|---|---|---|---|
| Amount of goods imported (in millions of dollars) | 5435 | 6547 | 7435 | 8250 | 9188 | 9341 | 9525 | 10367 |

Source: http://www.ita.doc.gov/td/industry/otea/usfth/aggregate/HL00T11.html

**Solution** Using the same set up as in Example 2.13 we find the logarithmic regression model to be $y = 1041.33 + 4007.79(\ln x)$ with $R^2 = 0.98769$, as shown below.

**Figure 2.9** Modeling Applet for Example 2.14




To predict the amount of imported goods in the year 2005 we input $x = 15$ into the regression calculator and find $y = 11{,}894.63$ million dollars. ◈

**Example 2.15** The table below represents the average price (in dollars) ranchers in the United States received per head of cattle from 1996 to 2001. Find the best fitting model for this data set and use the model to predict the average price per head in 1993.

| Year | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 |
|---|---|---|---|---|---|---|
| Average price paid per head of cattle (in dollars) | 503 | 525 | 603 | 594 | 683 | 725 |

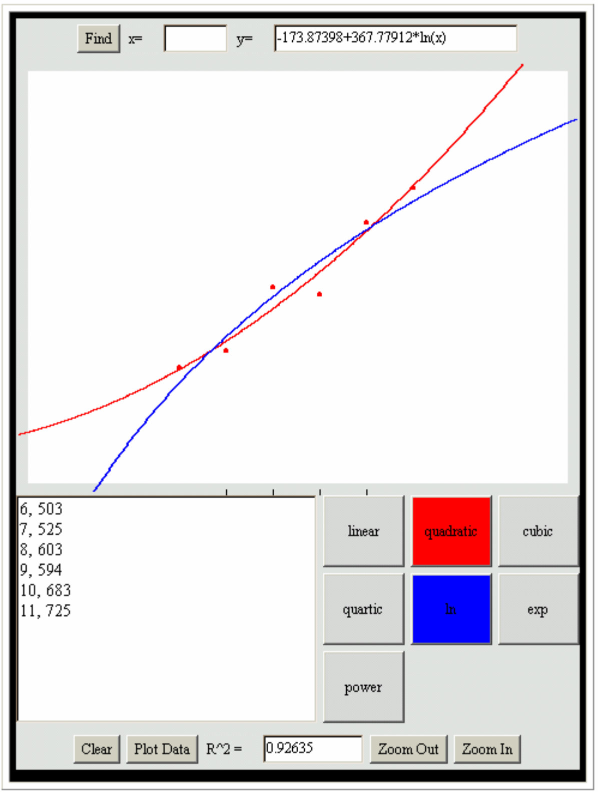
Source: http://www.usda.gov/nass/pubs/agr01/acro01.htm , hog/cattle/sheep

**Solution** Let $x$ be the number of years after 1990 and $y$ be the dollar amount paid per head of cattle. Using the regression calculator we get the following equations and $R^2$ values.

| Regression Model | Equation | $R^2$ |
|---|---|---|
| Exponential | $y = 319.14(1.08)^x$ | 0.95136 |
| Logarithmic | $y = -173.87 + 367.78(\ln x)$ | 0.92635 |
| Linear | $y = 45x + 223$ | 0.94724 |
| Quadratic | $y = 2.57x^2 + 1.29x + 401.29$ | 0.95383 |
| Cubic | $y = 0.37x^3 - 6.87x^2 + 79.69x + 189.73$ | 0.95407 |
| Power | $y = 164.68\left(x^{0.61}\right)$ | 0.93917 |

By comparing the $R^2$ values we see that the exponential, quadratic, and cubic models have the highest value. Since the cubic value is not significantly higher than the quadratic we can eliminate the cubic model because the difference in the $R^2$ value does not justify the more complicated model. Thus, we must decide between the exponential and quadratic model by studying their overall shape as seen in Figure 2.10.

**Figure 2.10** Modeling Applet for Example 2.15




Both graphs extend to positive infinity with the same general shape, but from 0 to 5 the graphs have slightly different characteristics. It appears that the exponential model has the same general shape as the data from [0, 5] whereas the quadratic model tends to deviate from the data's shape. It is for this reason that the exponential model would best fit this data set. Now using this model to predict the price per head of cattle in 1993 we need to find $y$ when $x = 3$. Using the regression calculator we find $y \approx \$399$. ◈

# Sample Quiz

**Question 2.1** Determine whether the following table represents an exponential function by calculating successive *f(x)* ratios. If it represents an exponential function, give the value of the base and state whether it is a growth or decay function.

| *x* | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| *f(x)* | 42 | 51 | 62 | 75 | 90 | 109 | 133 |

**Question 2.2** Solve $4^x - 13 \cdot 2^x = -36$ for *x*.

**Question 2.3** How much money should Dave deposit into an account that earns 5.5% annual interest compounded monthly if he wants to have $30,000 in the account 18 years from now?

**Question 2.4** After some research, you found two investment options for your $18,000 graduation gift. Bank Two offers 8% compounded semiannually and Bank One offers 7.5% compounded monthly. What is the difference in value in these two options at the end of 4 years?

**Question 2.5** Find the amount in an account after 7 years if $3,000 is compounded continuously at 5.25%.

**Question 2.6** Solve $\ln(\ln 2x) = 0$ for *x*.

**Question 2.7** The number of DVD players supplied to an electronics store is given by the equation $S(p) = 150e^{0.004p}$ where *p* is the price in dollars. If the store is supplied 215 DVD players, at what price should they sell them?

**Question 2.8** Given $\log_b 5 = 2.3219$, $\log_b 3 = 1.5850$, and $\log_b 7 = 2.8074$, find $\log_b\left(\frac{21}{25}\right)$.

**Question 2.9** The following table represents the assets (in billions of dollars) of FDIC insured Commercial Banks (*y*) in the United States from 1987 to 2001 (*x*). Find the best fitting exponential and logarithmic model, along with their $R^2$ values, and then determine which of the two would be the better model.

| *x* | 1987 | 1988 | 1989 | 1990 | 1991 | 1992 | 1993 | 1994 |
|---|---|---|---|---|---|---|---|---|
| *y* | 2,913 | 3,056 | 3,207 | 3,361 | 3,377 | 3,438 | 3,569 | 3,893 |
| *x* | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | |
| *y* | 4,171 | 4,397 | 4,771 | 5,181 | 5,469 | 5,983 | 6,360 | |

Source: http://www3.fdic.gov/sod/pdf/dsta_2001.pdf

**Question 2.10** The following table represents the assets (in billions of dollars) of FDIC insured Savings Institutions (*y*) in the United States from 1987 to 2001 (*x*). Find the best fitting model for this data and state why it is the best model.

| x | 1987 | 1988 | 1989 | 1990 | 1991 | 1992 | 1993 | 1994 |
|---|---|---|---|---|---|---|---|---|
| y | 1,441 | 1,556 | 1,512 | 1,317 | 1,161 | 1,078 | 1,004 | 999 |
| x | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | |
| y | 1,017 | 1,023 | 1,029 | 1,045 | 1,126 | 1,179 | 1,275 | |

Source: http://www3.fdic.gov/sod/pdf/dsta_2001.pdf

# Chapter 3 Limits and Continuity

# Chapter 4 Rates of Change

In this chapter we will investigate how fast one quantity changes in relation to another. The first type of change we investigate is the average rate of change, or the rate a quantity changes over a given interval. For example, if you have 15 minutes to arrive at a destination that is 10 miles away, you could calculate the average rate of change by dividing 10 miles by ¼ hour. Thus, you would need to travel 40 miles per hour to arrive at your destination on time. The second rate of change that we investigate, and actually begins our study of differential calculus, is instantaneous rate of change, or the rate a quantity changes at a given instant. Police officers parked on the side of a road, calculating the speed of cars with their radar gun is an example of instantaneous rate of change.

## 4.1 Average Rate of Change

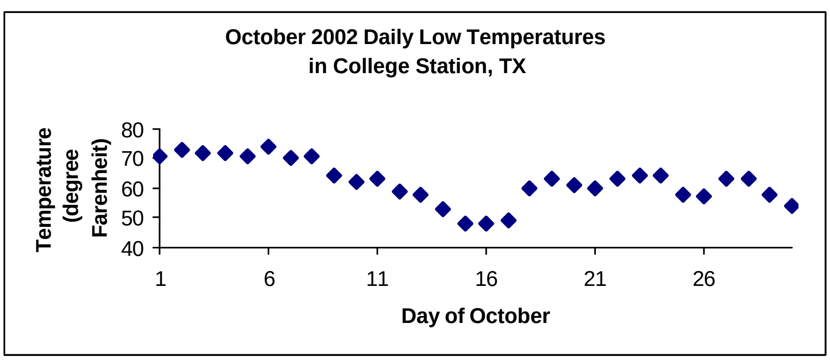
Table 4.1 below shows the daily low temperatures in College Station, Texas for the first 30 days in October 2002. This data is graphed as a scatter plot in Figure 4.1.

**Table 4.1** Daily low temperatures in October 2002 for College Station, TX.

| Day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Low (°F) Temperature | 71 | 73 | 72 | 72 | 71 | 74 | 70 | 71 | 64 | 62 | 63 | 59 | 58 | 53 | 48 |
| Day | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| Low (°F) Temperature | 48 | 49 | 60 | 63 | 61 | 60 | 63 | 64 | 64 | 58 | 57 | 63 | 63 | 58 | 54 |

*Source:* http://www.met.tamu.edu/met/osc/cll/oct02.htm

**Figure 4.1** Scatter Plot of Table 4.1 data

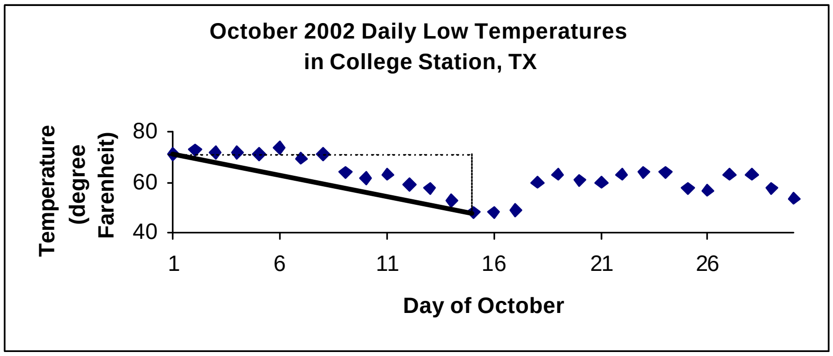


We can use this data to find the average change in temperature between the 1st and 15th day. Since the temperature was 71 degrees on the 1st of October and 48 degrees on the 15th of October we can conclude that over the 14-day period, the temperature changed 23 degrees. Representing this as a ratio we get

$$\frac{\text{change in temperature}}{\text{number of days passed}} = \frac{71-48}{1-15} = \frac{23}{-14} \approx -1.64 .$$

This tells us that the temperature dropped at an average rate of 1.64 degrees per day between the 1st and 15th day of October. Figure 4.2 below shows a graph of the average rate of change between 1st and 15th day.

**Figure 4.2** Average rate of change in low temperatures between 1st and 15th day of October.

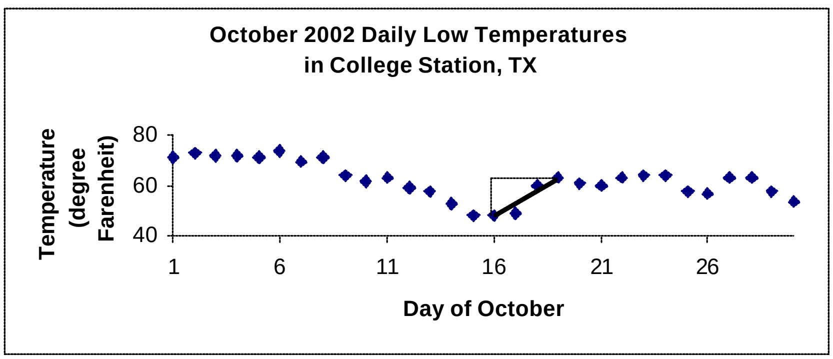



A similar calculation can be done for the change in temperature between the 16th and 19th day.

$$\frac{\text{change in temperature}}{\text{number of days passed}} = \frac{63-48}{19-16} = \frac{15}{3} = 5 .$$

Thus, the temperature increased at an average rate of 5 degrees per day between the 16th and 19th day of October. Figure 4.3 shows a graph of the average rate of change between the 16th and 19th days.

**Figure 4.3** Average rate of change in low temperatures between 16th and 19th day of October.




If we look closely at Figures 4.2 and 4.3 we notice that the way we calculate the average rate of change is the same way we calculate the slope of a line. As a result we define the average rate of change as follows.

**Average Rate of Change**
The average rate of change between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The units for the average rate of change are $\frac{\text{units of } y}{\text{units of } x}$.

**Example 4.1** The table below shows the hourly earnings (in dollars) for manufacturing plant employees in the United States from the years 1997 to 2001. Find the average rate of change in the hourly earnings from 1999 to 2001.

| Year (x) | 1997 | 1998 | 1999 | 2000 | 2001 |
|---|---|---|---|---|---|
| Hourly Earnings (y) | 13.17 | 13.49 | 13.90 | 14.37 | 14.83 |

*Source:* http://ftp.bls:gov/pub/suppl/empsit/ceseeb2.txt

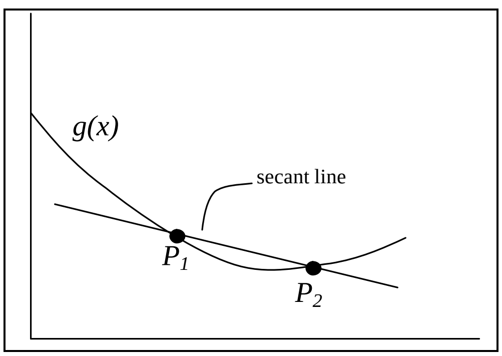
**Solution** The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{14.83 - 13.9}{2001 - 1999} = \frac{0.93}{2} = 0.465 .$$

Therefore, the average hourly earnings were increasing at a rate of 46.5 cents per year between 1999 and 2001. ◈

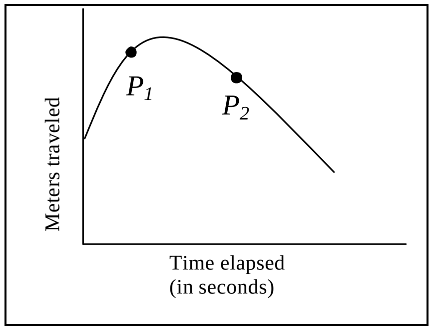
We can also find the average rate of change over a given interval when data is modeled by a function. The average rate of change between any two $x$ values is found by calculating the slope of the secant line, a line that intersects a curve at two points. The secant line that contains points $P_1$ and $P_2$ is shown in Figure 4.4 below.

**Figure 4.4** The secant line, or average rate of change, through points $P_1$ and $P_2$.




**Example 4.2** The graph in Figure 4.5 below represents the number of meters an object is above ground after $t$ seconds have elapsed. Find the average rate of change between $P_1$ (3, 25) and $P_2$ (8, 18).

**Figure 4.5** Graph of the distance an object traveled.




**Solution** The point $P_1$ (3, 25) means that after 3 seconds the object was 25 meters above the ground and $P_2$ (8, 18) means that after 8 seconds the object was 18 meters above the ground. The average rate of change is the slope of the secant line containing $P_1$ and $P_2$.

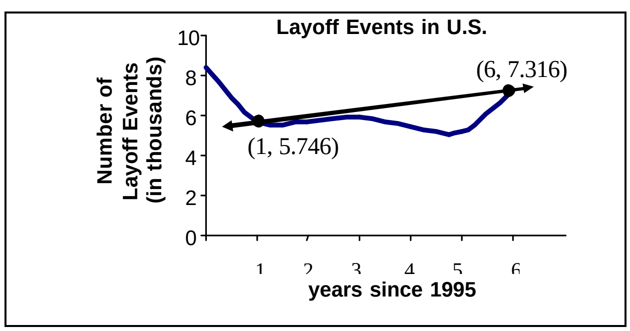
$$\frac{25-18}{3-8}=\frac{7}{-5}=-1.4\,.$$

This tells us that between the 3rd and 8th second, the object was falling at an average rate of 1.4 meters per second. ◈

**Example 4.3** The number of annual layoff events that occurred in the United States from 1996 to 2001 can be modeled by $f(x)=0.066x^4-0.8x^3+3.26x^2-5.2x+8.42,\ \ 1\le x\le 6$ where $x$ is the number of years since 1996 and *f(x)* is the number of thousand of events. Find the average rate of change in layoff events from 1996 to 2001. (*Source:* http://data.bls.gov/servlet/SurveyOutputServlet)

**Solution** Since 1996 corresponds to $x$ = 1 and 2001 corresponds to $x$ = 6, the number of thousands of layoff events in 1996 is $f(1)$ and the number of thousands of layoff events in 2001 is $f(6)$. Figure 4.6 shows a graph of $f(x)$ and the secant line that contains the points at $x$ = 1 and $x$ = 6.

**Figure 4.6** Graph of f(x) and the secant line that contains the points at $x$ = 1 and $x$ = 6.



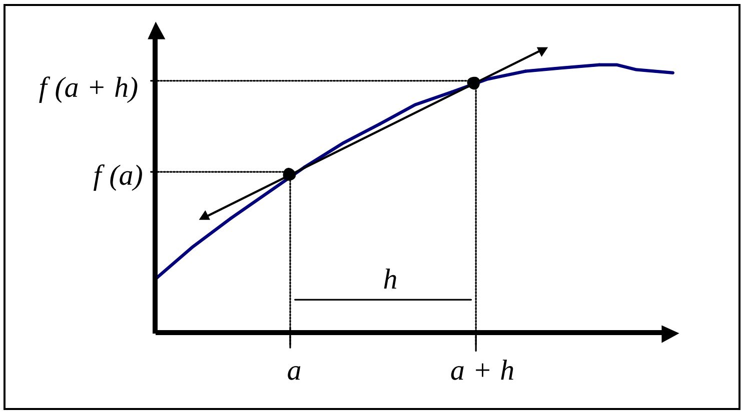

The slope of the secant line is

$$\frac{\Delta y}{\Delta x}=\frac{f(6)-f(1)}{6-1}=\frac{7.316-5.746}{6-1}=\frac{1.57}{5}=0.314$$

This means that the number of layoff events was increasing at a rate of 0.314 thousand, or 314, events per year from 1996 to 2001. ◈

It is useful to derive a general formula that will calculate the slope of the secant line for any function *f(x)*. To do this let $a$ and $a+h$ be two points on a continuous function *f(x)* such that $a < a+h$ as shown in Figure 4.7 below. (Notice that $h$ is the distance between the two points $a$ and $a+h$.)

**Figure 4.7** Secant line on $f(x)$




The slope of the secant line that contains (*a*, *f(a)*) and $(a+h,\ f(a+h))$ is

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}.$$

The formula above is known as the difference quotient.

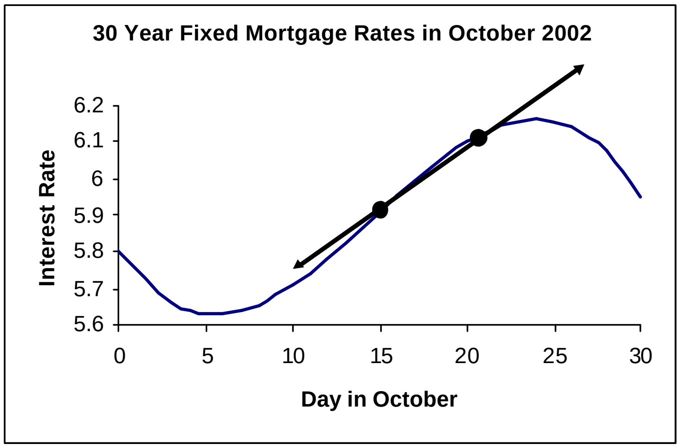
**Example 4.4** The 30-year mortgage rates during the month of October 2002 can be modeled by $f(x) = -0.00017x^3 + 0.0075x^2 - 0.067x + 5.8$ where *x* represents the day in October and *f(x)* represents the interest rate. Use the difference quotient to calculate the average rate of change in 30 year fixed mortgage rates from October 15th to October 21st.

**Solution** Since October 15th is represented by *x* = 15 and October 21st is represented by *x* = 21 we know that $h = 21 - 15 = 6$. Using the difference quotient to find the slope of the secant line we get

$$m_{\text{sec}} = \frac{f(a+h)-f(a)}{h} = \frac{f(15+6)-f(15)}{6} = \frac{f(21)-f(15)}{6} = \frac{6.1261-5.9088}{6} = 0.0362$$

So between October 15th and October 21st, the interest rates were changing at an average rate of 0.0362 percent per day as shown in Figure 4.8 below. ◈

**Figure 4.8** Average rate of change of fixed mortgage rates from the 15th to the 20th of October.



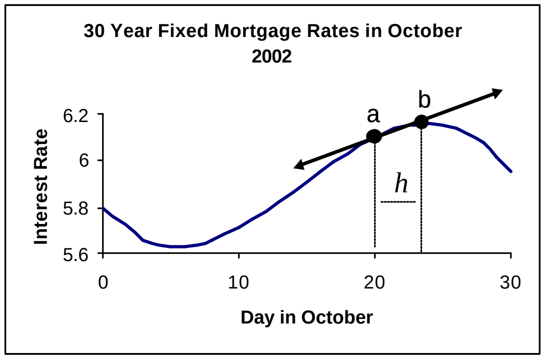

(Source http://www.bankrate.com/ust/subhome/mtg_m1.asp)

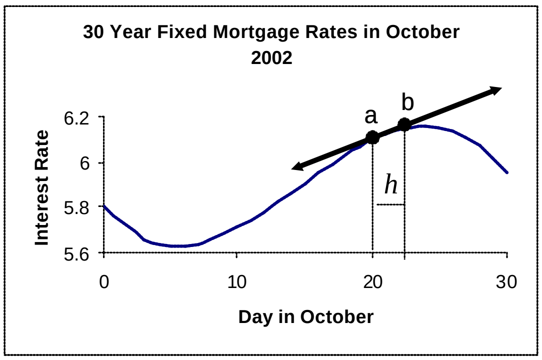
## 4.2 Instantaneous Rate of Change

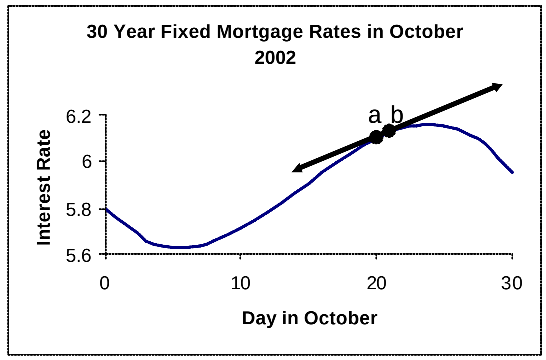
The average rate of change is a good calculation to use if we are looking for the rate of change over an interval. If, however, we want to find how the interest rates were changing on the 20th of October, calculating the slope of the secant line is no longer possible because the 20th of October is represented by a single point. Thus, we need the slope of the line that touches the graph at $x = 20$. This type of line is called a tangent line. We can find an approximation of the slope of the tangent line by calculating the slopes of secant lines that are close to $x = 20$ as shown in Figure 4.9.

**Figure 4.9** Slopes of secant lines where point b is getting closer and closer to point a (*h* is getting smaller).

(Source http://www.bankrate.com/ust/subhome/mtg_m1.asp)

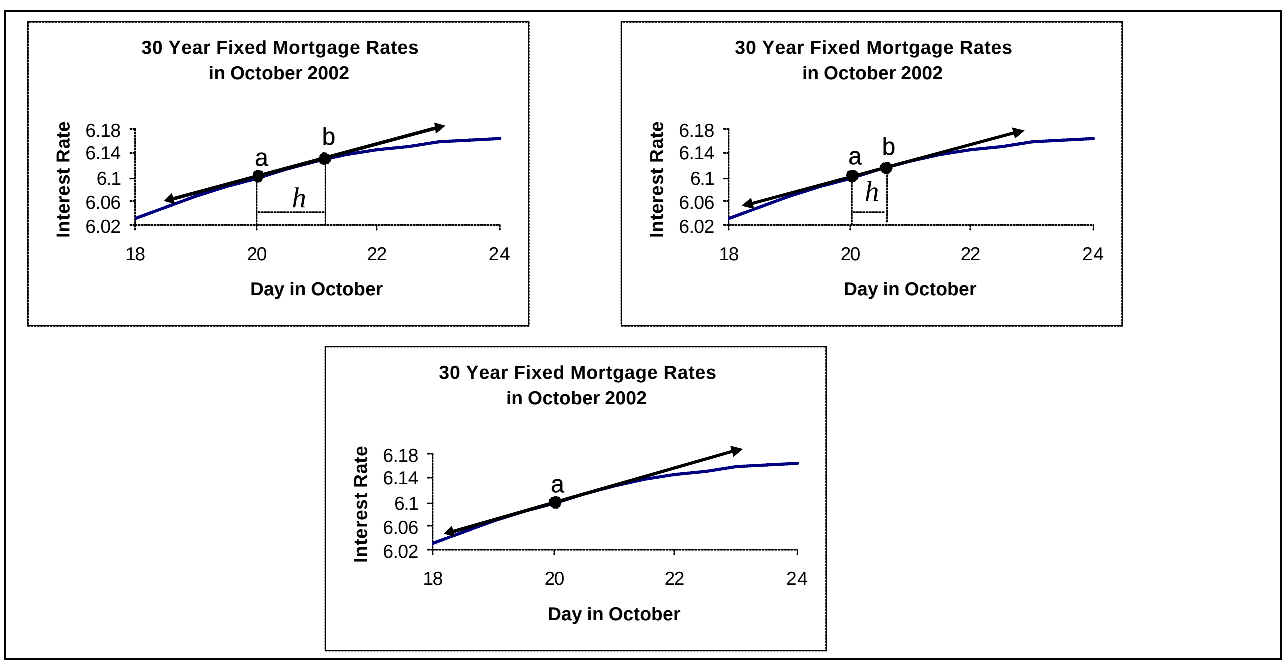



If we zoom in closer to point **a**, we can see, as shown in Figure 4.10 below, that the slope of the secant lines are approximately the same as the slope of tangent line at **a**.

**Figure 4.10** Graphs of the secant lines when we zoom in around point **a** and the graph of the tangent line at point **a**.



As the distance between **a** and **b** decreases, that is as $h$ approaches zero ( $h \to 0$ ), the slopes of the secant lines approach the slope of the line tangent at $x = a$ , as shown above in Figures 4.9 and 4.10. The slope of the tangent line at $x = a$ gives the rate of change in the mortgage rates at the instant $x = a$ , therefore we call the slope of the tangent line the instantaneous rate of change of *f(x)* at $x = a$ . This leads us to the following formula.

> If $f(x)$ is a continuous function, the rate of change at a single point $x = a$ , commonly called the instantaneous rate of change at *a* or the slope of the line tangent to $f(x)$ at *a*, can be found by computing
>
> $$f'(a) = m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
>
> provided the limit exists.

**Note:** The notation $f'(a)$ , read "f prime of a", is commonly use to denote the instantaneous rate of change of $f(x)$ at $x = a$ .

**Example 4.5** Janice's Jewelry store has found that the amount of profit her store makes each day after Christmas can be modeled by $P(x) = -4x^2 + 40x - 60$ where *x* is the number of days after December 25th and *P(x)* is the profit, in hundreds of dollars. Find the instantaneous rate of change in profit for Janice's Jewelry store for the following days and interpret each answer.

**a.** the 3rd day after Christmas.

**b.** the 6th day after Christmas.

**Solution**

**a.** The 3rd day after Christmas is represented by $x = 3$ , thus we substitute $a = 3$ into the instantaneous rate of change formula above we get

$$m_{\tan} = P'(3) = \lim_{h \to 0} \frac{P(3+h) - P(3)}{h}$$

It will be easiest if we first find $P(3+h)$ . In doing so we get

$$\begin{aligned} P(3+h) &= -4(3+h)^2 + 40(3+h) - 60 = -4(9+6h+h^2) + 120 + 40h - 60 \\ &= -36 - 24h - 4h^2 + 40h + 60 = -4h^2 + 16h + 24 \end{aligned}$$

Now we need to find $P(3)$

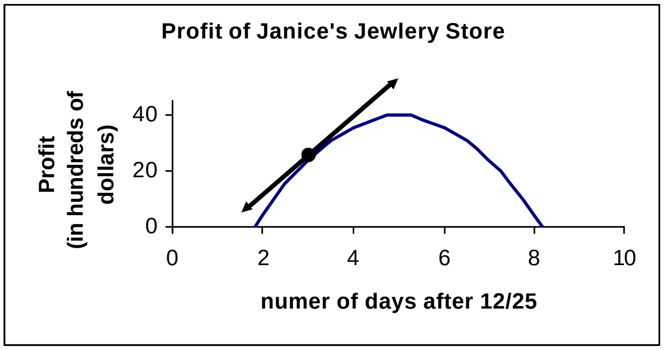
$$P(3) = -4(3)^2 + 40(3) - 60 = -4(9) + 120 - 60 = 24$$

Substituting these expressions into the formula above we get

$$m_{\tan} = \lim_{h \to 0} \frac{P(3+h) - P(3)}{h} = \lim_{h \to 0} \frac{-4h^2 + 16h + 24 - 24}{h} = \lim_{h \to 0} \frac{h(-4h+16)}{h} = \lim_{h \to 0}(-4h+16) = 16$$

This means that the profit was increasing at a rate of $1600 per day on the 3rd day after Christmas. Figure 4.11 below shows a graph of this instantaneous rate of change.

**Figure 4.11** A graph of the tangent line, or instantaneous rate of change, at $x = 3$, which represents the third day after Christmas.




**b.** The 6th day after Christmas is represented by $x = 6$, thus we substitute $a = 6$ into the instantaneous rate of change formula above we get

$$m_{\tan} = P'(6) = \lim_{h\to 0}\frac{P(6+h) - P(6)}{h}$$

First find $P(6+h)$.

$$P(6+h) = -4(6+h)^2 + 40(6+h) - 60 = -4(36 + 12h + h^2) + 240 + 40h - 60$$
$$= -144 - 48h - 4h^2 + 180 + 40h = -4h^2 - 8h + 36$$

Then find $P(6)$.

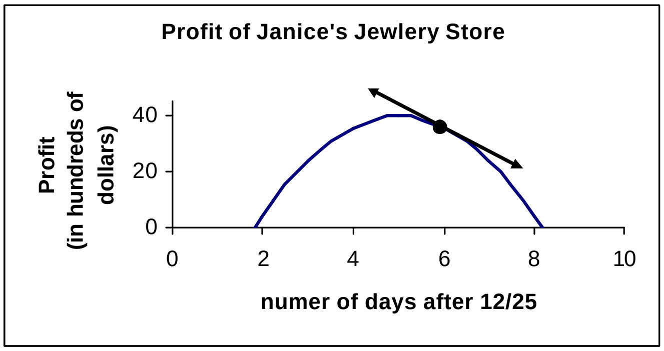
$$P(6) = -4(6)^2 + 40(6) - 60 = -4(36) + 240 - 60 = 36$$

Substituting this into the formula above we get

$$m_{\tan} = P'(6) = \lim_{h\to 0}\frac{P(6+h) - P(6)}{h} = \lim_{h\to 0}\frac{-4h^2 - 8h + 36 - 36}{h}$$
$$= \lim_{h\to 0}\frac{h(-4h - 8)}{h} = \lim_{h\to 0}(-4h - 8) = -8$$

This means that the profit was decreasing at a rate of 8 hundreds dollars per day on the 6th day after Christmas. A graph of this is shown in Figure 4.12 below. ◈

**Figure 4.12** A graph of the tangent line, or instantaneous rate of change, at $x=6$, which represents the 6th day after Christmas.




**Example 4.6** Find the equation of the line tangent to

$$f(x)=\frac{1}{x} \text{ at } x=1.$$

**Solution** The tangent line will have slope $m_{\tan}=f'(1)$ and will contain the point $(1, f(1))=(1,1)$. Thus, using the point-slope formula, the equation of the line tangent to

$$f(x)=\frac{1}{x} \text{ at } x=1$$

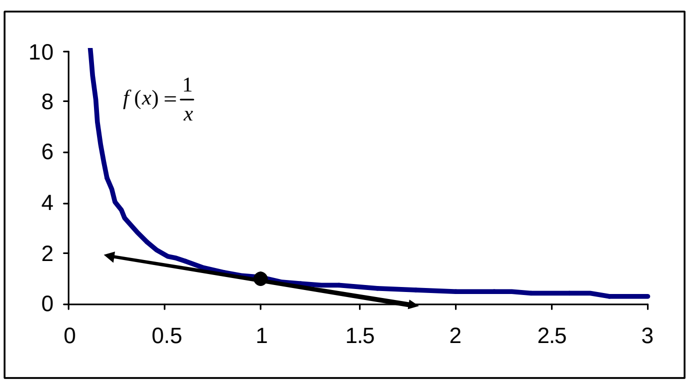
will have the form $y-f(1)=f'(1)(x-1)$. The slope of the tangent line is computed as shown below.

$$m_{\tan}=f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}=\lim_{h\to 0}\frac{\frac{1}{1+h}-\frac{1+h}{1+h}}{h}=\lim_{h\to 0}\frac{\frac{-h}{1+h}}{h}$$

$$=\lim_{h\to 0}\frac{-h}{1+h}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{-1}{1+h}=-1$$

The equation of the tangent line is $y-1=-1(x-1)$ which is the same as $y=-x+2$. A graph of this is shown below in Figure 4.13. ◈

**Figure 4.13** A graph of the line tangent to $f(x)=1/x$ at $x=1$.



**Example 4.7** Find the equation of the line tangent to $f(x)=\sqrt{x+1}$ at $x=3$.

**Solution** The slope of the line tangent to $f(x)$ is

$$m_{\tan} = f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{\sqrt{3+h+1}-\sqrt{4}}{h} = \lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}$$

To find this limit we must multiply the numerator and denominator by the conjugate of $\sqrt{h+4}-2$ which is $\sqrt{h+4}+2$.

$$m_{\tan} = \lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}\cdot\frac{\sqrt{h+4}+2}{\sqrt{h+4}+2} = \lim_{h\to 0}\frac{h+4-4}{h\left(\sqrt{h+4}+2\right)} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{h+4}+2\right)}$$

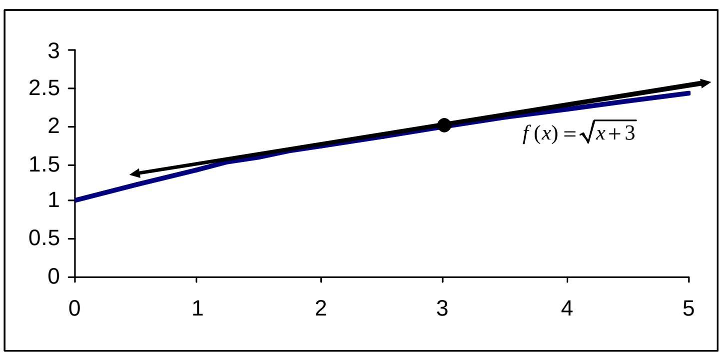
$$= \lim_{h\to 0}\frac{1}{\sqrt{h+4}+2} = \frac{1}{2+2} = \frac{1}{4} = 0.25$$

The equation of the line tangent to $f(x)$ at $x=3$ is

$$y - f(3) = f'(3)(x-3)$$
$$y - 2 = 0.25(x-3)$$
$$y = 0.25x - 0.75 + 2 = 0.25x + 1.25$$

A graph of this is shown in Figure 4.14 below. ◈

**Figure 4.14** A graph of the line tangent to $f(x)=\sqrt{x+1}$ at $x=3$.




**Example 4.8** Arlen's Air Service provides airline service for private individuals. The cost for flying from Austin, TX to Denver, CO can be modeled by $C(x)=x^3-12x^2+36x+50$ where $x$ is the number of round trips made and $C(x)$ is the cost of the trip in hundreds of dollars. Arlen has found that the instantaneous rate of change of $C(x)$ at any point $x=a$ is $C'(a)=3a^2-24a+36$. Find slope of the line tangent to $C(x)$ at $x=8$ and interpret your answer.

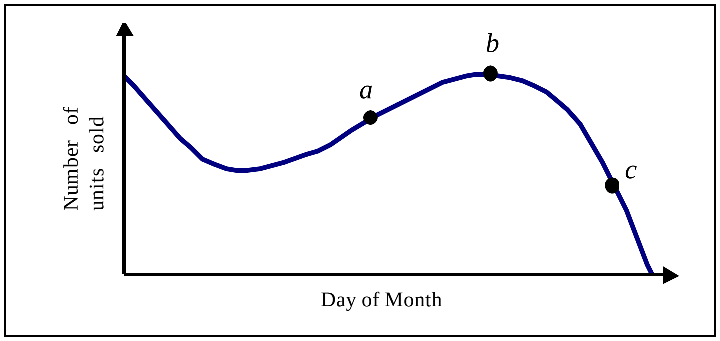
**Solution** We are given $C'(a)=3a^2-24a+36$, thus the slope of the line tangent to $C(x)$ at $x=8$ is found by evaluating $C'(8)$.

$$m_{\tan} = C'(8) = 8^3 - 12(8)^2 + 36(8) + 50 = 82 .$$

We conclude that the cost for flying from Austin to Denver on the 8th round trip was increasing at a rate of $8200 per round trip.

**Example 4.9** In Figure 4.15 below, $g(x)$ represents the number of units a company sells in one month and $x$ represents the day of the month. Determine whether the slope of the tangent lines at points *a*, *b*, and *c* are positive, negative, zero, or undefined. Interpret your answers.

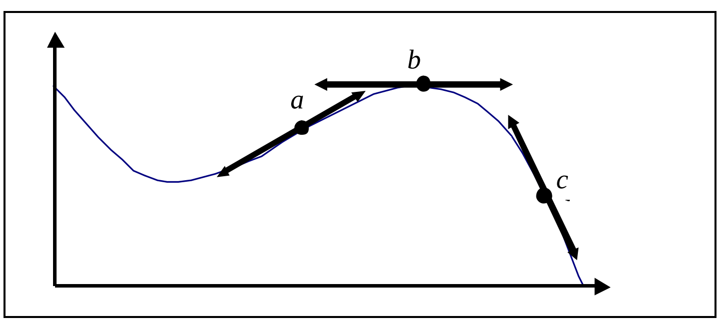
**Figure 4.15** Graph for Example 4.9




**Solution** The tangent lines to each point are shown in Figure 4.16 below and we make the following conclusions.

- The tangent line at *a* is positive which means that on the $a^{th}$ of the month the rate of change of sells was increasing.
- The slope of the tangent line at *b* is zero, therefore, on the $b^{th}$ day of the month the rate of change of sells was not changing.
- The slope of the tangent line at *c* is negative, therefore, on the $c^{th}$ day of the month the rate of change of sells were decreasing. ◈

**Figure 4.16** Graph for Example 4.9



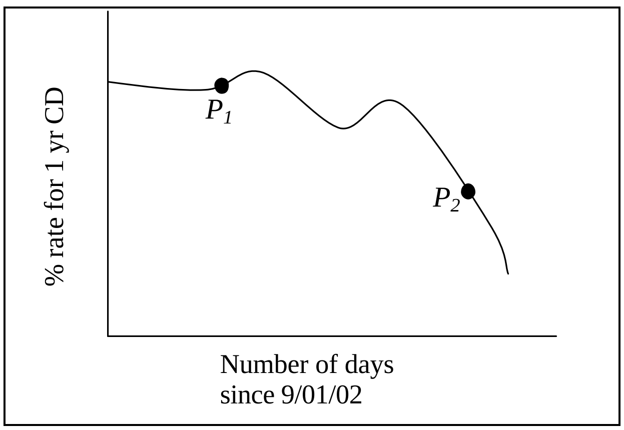
# Sample Quiz

**Question 4.1** The table below shows the number of unemployed Texans in the San Marcos\Austin area from January 2001 to December 2001. Let *x* represent the month, with $x = 1$ representing January, and *y* represent the number of unemployed Texans. Find the average rate of unemployment from April to November.

| Month | Jan | Feb | Mar | Apr | May | June | July | Aug | Sept | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of Unemployed | 16189 | 17970 | 20010 | 20907 | 25617 | 34055 | 34257 | 35222 | 36212 | 35453 | 37097 | 35245 |

*Source:* http://data.bls.gov

**Question 4.2** The given graph represents the annual percentage rates for a 1 year certificate of deposit (CD) from September 1, 2002 to November 1, 2002. Use the graph to find the average rate of change for a 1 year CD from $P_1$ (16, 2.37) to $P_2$ (68, 2.1). *Source:* http://www.bankrate.com




**Question 4.3** The number of annual layoff events that occurred in the United States from 1996 to 2001 can be modeled by $f(x) = 0.066x^4 - 0.8x^3 + 3.26x^2 - 5.2x + 8.42, \quad 1 \le x \le 6$ where *x* is the number of years since 1996 and $f(x)$ is the number of thousand of events. Find the average rate of change in layoff events from 1997 to 2000.

**Question 4.4** Given $f(x) = 3/x$, find the instantaneous rate of change of $f(x)$ at $x = 2$.

**Question 4.5** Given $f(x) = \sqrt{x-2}$, find the slope of the line tangent to $f(x)$ at $x = 3$.

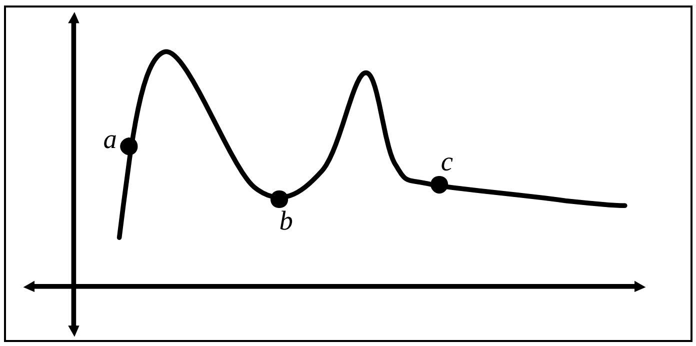
**Question 4.6** Mike's Mean Machine Shop sells motorcycles and has determined that $R(x) = 0.02x^2 + x$ models the amount of revenue (in thousands of dollars) after selling *x* motorcycles. Find and interpret the instantaneous rate of change of the shop's revenue at $x = 6$.

**Question 4.7** Find the equation of the line tangent to $f(x) = 3x^2 - 1$ at $x = -2$.

**Question 4.8** Find the equation of the line tangent to $g(x) = \sqrt{x}$ at $x = 9$.

**Question 4.9** Kathryn's Pottery shop has determined that $g(x) = 0.45x^2 - 10$ models the number of pieces of pottery she can make in $x$ hours. Find the slope of the line tangent to $g(x)$ at $x = 16$ and interpret your answer.

**Question 4.10** Use the graph below to determine whether the slope of the tangent lines at points *a*, *b*, and *c* are positive, negative, zero, or undefined.



# Chapter 5   Derivatives

# Chapter 6   Curve Sketching

# Chapter 7 Optimization

## 7.1 Finding Absolute Extrema

## 7.2 Maximization Problems

## 7.3 Minimization Problems

## Sample Quiz

# Chapter 8 Indefinite Integrals

# Chapter 9   Definite Integrals

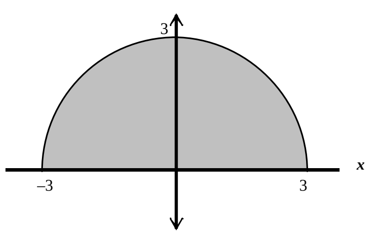
In the previous chapter we found how to take an antiderivative and investigated the indefinite integral.  In this chapter the connection between antiderivatives and definite integrals is established as we try to solve one of the most famous problems in mathematics, finding the area under a given curve.

## 9.1 Approximating Area Under a Curve

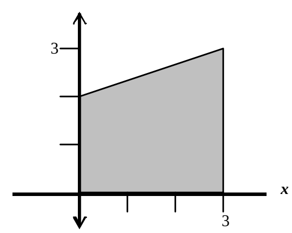
When it comes to finding the area of basic geometric shapes such as circles, squares, rectangles, triangles, and trapezoids, we can rely on geometric formulas to calculate the area.

**Example 9.1**   Find the area of the shaded regions.

**a.**




**b.**



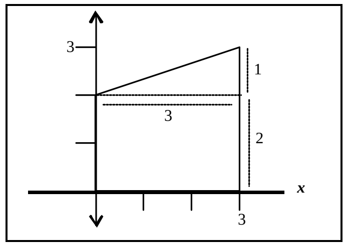

**Solution**

**a.** The shaded region is half of a circle with a radius of 3.  Thus, we will use the formula for the area of a circle ( $A = \pi r^2$ ), multiply it by 0.5 and use $r = 3$ .

$$A_{region} = .5\left(\pi (3)^2\right) = .5(9\pi) = 4.5\pi$$

**b.** The shaded region is made up of a rectangle $\left(A = lw, l = 3, w = 2\right)$ and a triangle $\left(A = \frac{1}{2}bh, b = 3, h = 1\right)$.  If we add the areas of the two geometric shapes, we will have found the area of the shaded region.

**Figure 9.1** Graph for Example 9.1a

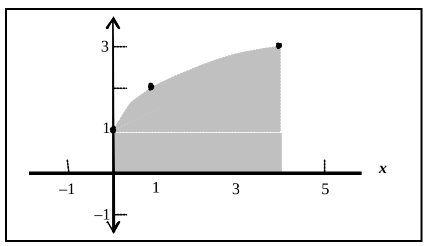



$$A_{\text{shaded}} = A_{\text{rectangle}} + A_{\text{triangle}} = (lw) + \left(\frac{1}{2}bh\right) = (3\cdot 2) + \left(\frac{1}{2}\cdot 3\cdot 1\right) = 6 + \frac{3}{2} = \frac{15}{2}$$

The area of the trapezoid is therefore 7.5 square units. ◈

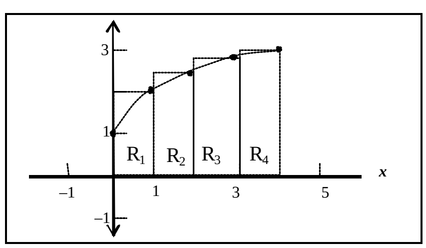
Finding the area between the *x*-axis and a curve $f(x)$ on a given interval is a bit more challenging if the region formed is not a "basic" geometric shape. For example, the area under the curve $f(x) = \sqrt{x} + 1$ on [0 ,4] forms the shape shown in Figure 9.2.

**Figure 9.2** Graph of $f(x) = \sqrt{x} + 1$.




We can see from the figure that the area between the *x*-axis and $f(x)$ is not a shape that has a familiar formula for finding the area. When this occurs, we use rectangles to approximate the area of the region. If we draw four rectangles, as seen in Figure 9.3, we can sum up the area of the rectangles ($R_1 + R_2 + R_3 + R_4$) and obtain an approximation of the area under the curve.

**Figure 9.3** Graph of $f(x) = \sqrt{x} + 1$ with 4 right endpoints[H].




[H] The rectangles have been constructed such that the right endpoint, $x_i$, of the interval touches the curve. Because of this, we call these rectangles right endpoint rectangles

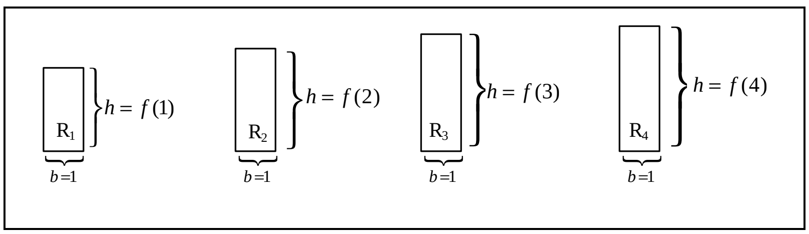
We will, however, find an overestimate of the area because the rectangles extend above the curve. Nonetheless, we will have some idea of the area under the curve.

To find the area of each rectangle in Figure 9.3 we need to find the base and height of each rectangle. The base of each rectangle, $\Delta x$, is found by taking the length of the given interval, $x_{max} - x_{min}$ and dividing it by the number of rectangles constructed, $n$. This leads to the following calculation.

$$\Delta x = \frac{x_{max} - x_{min}}{n} = \frac{4}{4} = 1.$$

The height of each rectangle is the value of the function from the right end of each interval. Figure 9.4 below shows the dimensions of each rectangle.

**Figure 9.4** Dimensions of $R_1$, $R_2$, $R_3$, and $R_4$.



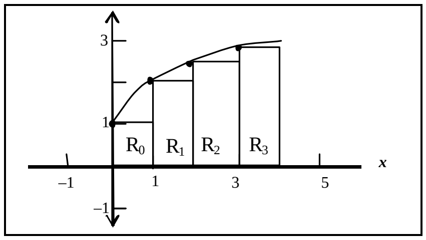

The sum of the rectangles are found in the table below. The area, *A*, is the base times the height.

| Rectangle # | Base, $\Delta x$ | Right Endpt, $x_i$ | Height, $f(x_i)$ | $A = \Delta x \cdot f(x_i)$ |
|---|---|---|---|---|
| $R_1$ | 1 | 1 | $f(1) = \sqrt{1} + 1 = 2$ | (1)(2) = 2 |
| $R_2$ | 1 | 2 | $f(2) = \sqrt{2} + 1 \approx 2.41$ | (1)(2.41) = 2.41 |
| $R_3$ | 1 | 3 | $f(3) = \sqrt{3} + 1 \approx 2.73$ | (1)(2.73) = 2.73 |
| $R_4$ | 1 | 4 | $f(4) = \sqrt{4} + 1 = 2 + 1 = 3$ | (1)(3) = 3 |
| | | | Total Area (Right) | 10.14 |

Since this is an overestimate, the area under the curve is less then 10.14 units.

We can also approximate the area under the curve using left endpoint rectangles as shown in figure 9.4. This approximation will give us an underestimate because the rectangles do not fill the entire area under the curve.

**Figure 9.5** Graph of $f(x) = \sqrt{x} + 1$ with 4 left endpoint rectangles. (rename rectangles!!)




The base of each rectangle is still 1 unit but the height of each rectangle is the value of the function from the left end of each interval. The sum of the rectangles is found in the table below.

| Rectangle # | Base $\Delta x$ | Left Endpt, $x_i$ | Height, $f(x_i)$ | $A = \Delta x \cdot f(x_i)$ |
|---|---|---|---|---|
| $R_1$ | 1 | 0 | $f(0) = \sqrt{0} + 1 = 1$ | (1)(1) = 1 |
| $R_2$ | 1 | 1 | $f(1) = \sqrt{1} + 1 = 2$ | (1)(2.41) = 2.41 |
| $R_3$ | 1 | 2 | $f(2) = \sqrt{2} + 1 \approx 2.41$ | (1)(2.73) = 2.73 |
| $R_4$ | 1 | 3 | $f(3) = \sqrt{3} + 1 \approx 2.73$ | (1)(3) = 3 |
| | | | Total Area (Left) | 9.14 |

Thus, the area must be greater than 9.14 units.

In general, we can find an approximation for the area under a continuous curve $f(x)$ on $[a, b]$ by drawing $n$ equally spaced right (or left) endpoint rectangles under the curve and then finding the sum of the area of the rectangles. If $\Delta x$ is the width of each rectangle and $x_i$ an endpoint where $x_0 = a$ and $x_n = b$, then the sum of the area of $n$ rectangles is

$$A = \begin{pmatrix} \text{area of } 1^{\text{st}} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of } 2^{\text{nd}} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of } 3^{\text{rd}} \\ \text{rectangle} \end{pmatrix} + \ldots + \begin{pmatrix} \text{area of } n^{\text{th}} \\ \text{rectangle} \end{pmatrix}$$

For right endpoint rectangles the sum of the area rectangles can be denoted as [H]

$$\text{Total Area (Right)} = \sum_{i=1}^{n} \Delta x\ f(x_i) = \Delta x\ f(x_1) + \Delta x\ f(x_2) + \Delta x\ f(x_3) + \ldots + \Delta x\ f(x_n)$$

For left endpoint rectangles, the sum of the area of the rectangles can be denoted as

$$\text{Total Area (Left)} = \sum_{i=0}^{n-1} \Delta x\ f(x_i) = \Delta x\ f(x_0) + \Delta x\ f(x_1) + \Delta x\ f(x_2) + \ldots + \Delta x\ f(x_{n-1})$$

[H] The symbol $\sum$ is the notation for "the sum of "

**Example 9.2** Approximate the area under the curve $f(x)=x^2+2$ on $[0,2]$ using

**a.** 4 right endpoint rectangles

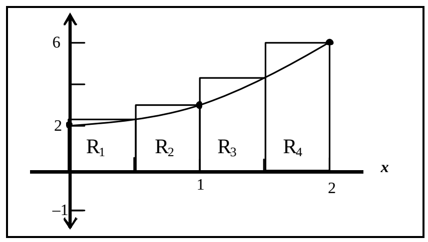
**b.** 8 left endpoint rectangles.

State if the estimate is an overestimate or an underestimate.

**Solution**

**a.** The graph of $f(x)=x^2+2$ on [0, 2] with the 4 right endpoint rectangles is shown in Figure 9.6.

**Figure 9.6** Graph of 4 right endpoint rectangles




The base of each rectangle is

$$\Delta x=\frac{b-a}{n}=\frac{2-0}{4}=\frac{1}{2}=0.5$$

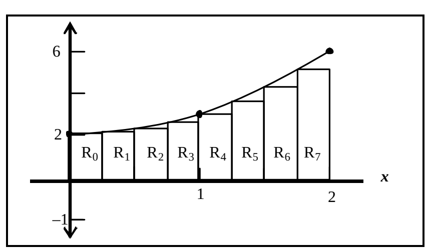
and the height is $f(x_i)$ where $x_i$ is the right endpoint of each interval. The calculation below shows the sum of the areas of the four rectangles.

$$\begin{aligned}\text{Total Area (Right)}=\left(\sum_{i=1}^{4}\Delta x\ f(x_i)\right)&=0.5\cdot f(x_1)+0.5\cdot f(x_2)+0.5\cdot f(x_3)+0.5\cdot f(x_4)\\&=0.5\cdot f(0.5)+0.5\cdot f(1)+0.5\cdot f(1.5)+0.5\cdot f(2)\\&=0.5\left[(0.5)^2+2\right]+0.5\left[(1)^2+2\right]+0.5\left[(1.5)^2+2\right]+0.5\left[(2)^2+2\right]\\&=7.75\end{aligned}$$

Thus, the area under the curve is less than 7.75 square units.

**b.** The graph of $f(x)=x^2+2$ on [0, 2] with the 8 left endpoint rectangles is shown in Figure 9.7.

**Figure 9.7** Graph of 8 left endpoint rectangles. (rename rectangles!!)



The base of each rectangle is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4} = 0.25$$

and the height is $f(x_i)$ where $x_i$ is the left endpoint of each interval. The table below shows the sum of the areas of the eight rectangles.

$$\text{Total Area (Left)}=\left(\sum_{i=1}^{8} \Delta x \ f(x_i)\right)$$

$$= 0.5 \cdot f(x_1) + 0.5 \cdot f(x_2) + 0.5 \cdot f(x_3) + 0.5 \cdot f(x_4) + 0.5 \cdot f(x_5) + 0.5 \cdot f(x_6) + 0.5 \cdot f(x_7) + 0.5 \cdot f(x_8$$

$$= 0.5 \cdot f(0) + 0.5 \cdot f(0.25) + 0.5 \cdot f(0.5) + 0.5 \cdot f(0.75) + 0.5 \cdot f(1) + 0.5 \cdot f(1.25)$$

$$+ 0.5 \cdot f(1.5) + 0.5 \cdot f(1.75)$$

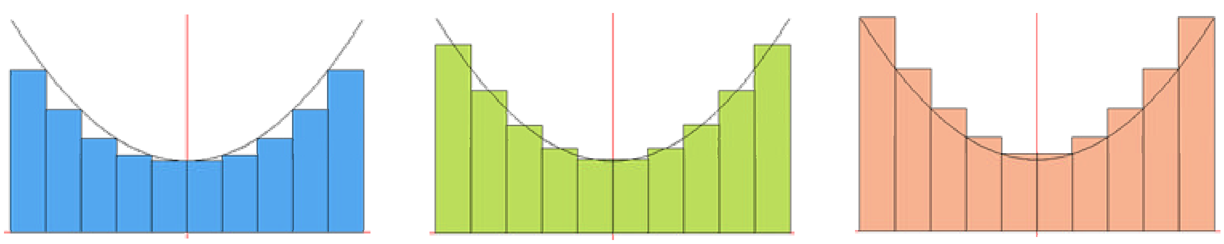
$$= 0.5\left[(0)^2 + 2\right] + 0.5\left[(0.25)^2 + 2\right] + \ldots + 0.5\left[(1.75)^2 + 2\right]$$

$$= 6.1875$$

Since these rectangles all lie below the curve, the estimate for the area under the curve is an underestimate. ◈

There are numerous methods of using rectangles to approximate the area under a curve. A few of the other methods are shown in Figure 9.8 below.

**Figure 9.8** Other methods to approximate $f(x) = x^2 + 2$ on $[-2,2]$.



**(a)** Lower Sum Method – all rectangles lie below the curve.

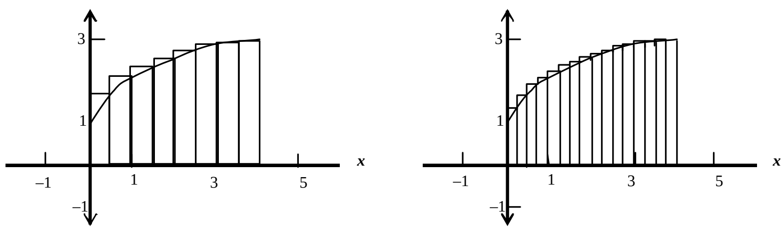
**(b)** Midpoint Method – the midpoint of all rectangles are touching the curve.

**(c)** Upper Sum Method – all rectangles lie above the curve.

# 9.2 Definite Integrals and the Fundamental Theorem of Calculus

The methods used in the previous section allow us to obtain a good approximation of the area under a curve, but can we make this approximation better? If we take thinner and thinner rectangles, we can make the approximation of the area under the curve more accurate. In Example 9.2 we found an approximation of 6.1875 square units for the area under the curve using 8 left endpoint rectangles. If we would have used 4 left endpoint rectangles our approximation would have been 5.75 square units. The approximation with 8 rectangles was more accurate simply because more rectangles were used. Compare the rectangles in Figure 9.9.

**Figure 9.8** Graphs of $f(x)=\sqrt{x}+1$ with 8 and 16 right endpoint rectangles.




It appears that the amount of excess area made by the 16 rectangles is considerably less than the excess area made by the 8 rectangles. One can imagine that the approximation would be even better if we could fit 100 rectangles or even 1000 rectangles under the curve. What if we had an infinite number of rectangles drawn under the curve? As one might hypothesize, the sum of an infinite number of rectangles does accurately find the area under a curve, and we represent the area under a curve using the definite integral.

**The Definite Integral**

For a continuous function *f* on the interval $[a,\ b]$[H] let $\Delta x = {}^{(b-a)}\!/_{n}$ and $x_i$ be the right endpoint of the *n* intervals. Then the definite integral of *f* is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\ \cdot f(x_i)$$

Some useful properties of definite integrals are listed in Table 9.1.

**Table 9.1** Properties of Definite Integrals

$$\int_a^b k\cdot f(x)\ dx = k\int_a^b f(x)\ dx$$

$$\int_a^b [f(x)\pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$$

$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx$$

[H] Note: "a" is referred to as the lower limit and "b" as the upper limit. Together, "a" and "b" are known as the limits of integration.

Using the definition of the definite integral the area in Figure 9.9 is represented as

$$\int_0^4 \left(\sqrt{x}+1\right)\, dx$$

**Example 9.3** Represent the area of the shaded regions from Example 9.1 as definite integrals.

**Solution**

**a.** $\int_{-3}^{3} \sqrt{9-x^2}\ \ dx$

**b.** $\int_0^3 (\tfrac{1}{3}x+2)\ \ dx$ ◈

From Example 9.1 (b) we found the area to be exactly $15/2$ and from Example 9.3 (b) we found that the area can be represented as a definite integral. We can put the two of these together and conclude

$$\int_0^3 (\tfrac{1}{3}x+2)\ \ dx = \frac{15}{2}.$$

Next we can connect the notion of an antiderivative and a definite integral. Take the antiderivative of $\tfrac{1}{3}x+2$,

$$F(x) = \frac{1}{3}\left(\frac{1}{2}x^2\right) + 2x + C = \frac{1}{6}x^2 + 2x + C$$

Note that $F(0) = C$ and $F(3)$ is

$$F(3) = \frac{1}{6}(3^2) + 2(3) + C = \frac{15}{2} + C$$

now find $F(3) - F(0)$,

$$F(3) - F(0) = \frac{15}{2} + C - C = \frac{15}{2}.$$

We can see that finding the antiderivative $F(x)$ of a function and then evaluating $F(b) - F(a)$ gives the exact area under the curve. This process is an important theorem in calculus known as the Fundamental Theorem of Calculus.

**The Fundamental Theorem of Calculus**
If $f$ is a continuous function defined on a closed interval $[a, b]$ and $F$ is an antiderivative of $f$, then

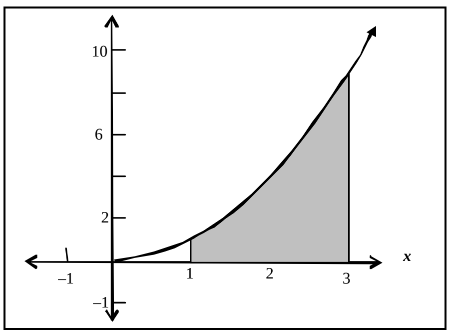
$$\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\Big]_a^b$$

**Example 9.4** Draw a geometric representation of each definite integral and then evaluate the definite integral using the Fundamental Theorem of Calculus.

**a.** $\int_1^3 x^2\ dx$ **b.** $\int_1^e \frac{1}{x} dx$

**Solution**

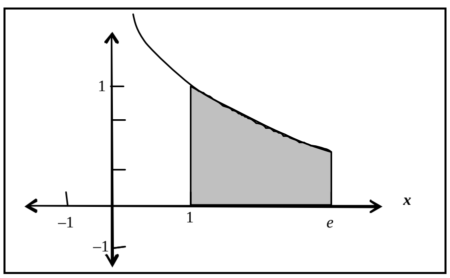
**a.** The shaded region in the graph below shows the geometric representation.




Use the Fundamental Theorem of Calculus to find the value of the definite integral.

$$\int_1^3 x^2\ dx = \frac{1}{3}x^3\Big]_1^3 = \frac{1}{3}(3^3) - \frac{1}{3}(1^3) = 9 - \frac{1}{3} = \frac{26}{3}$$

**b.** The shaded region in the graph below shows the geometric representation.



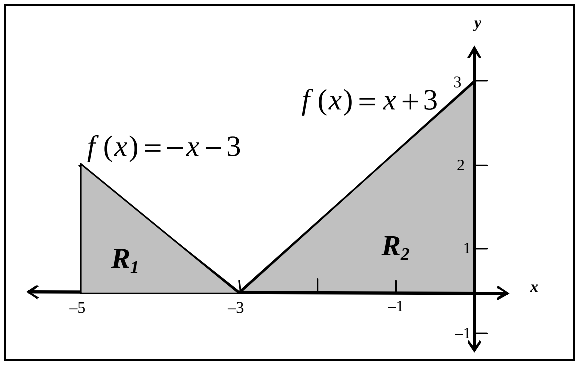

Use the Fundamental Theorem of Calculus to find the value of the definite integral.

$$\int_1^e \frac{1}{x} dx = \ln x\Big]_1^e = \ln e - \ln 1 = 1 - 0 = 1$$

**Example 9.5** Graph $f(x)=|x+3|$ and use the graph to find

$$\int_{-5}^{0}|x+3|\,dx\,.$$

**Solution** The graph of $f(x)=|x+3|$ is shown below.




To find $\int_{-5}^{0}|x+3|\,dx$, we need to write an integral that represents $\boldsymbol{R}_1$ and another to represent $\boldsymbol{R}_2$. This is necessary because $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are bounded by different functions.

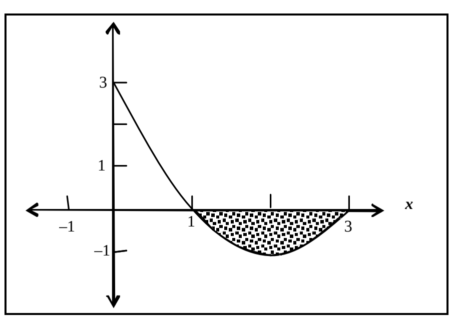
$$\int_{-5}^{0}|x+3|\,dx=\int_{-5}^{-3}-x-3\,dx+\int_{-3}^{0}x+3\,dx=-\frac{1}{2}x^2-3x\Big]_{-5}^{-3}+\frac{1}{2}x^2+3x\Big]_{-3}^{0}=2+4.5=6.5$$ ◈

All of the integrals we have considered thus far have been positive. That is the graphs of the functions lied strictly above the $x$ – axis. The next example demonstrates what happens when a shaded region lies strictly below the $x$ – axis.

**Example 9.6** Draw a geometric representation of

$\int_{1}^{3}(x^2-4x+3)\;dx$ then evaluate the definite integral using the Fundamental Theorem of Calculus.

**Solution** The shaded region in the graph below shows the geometric representation.



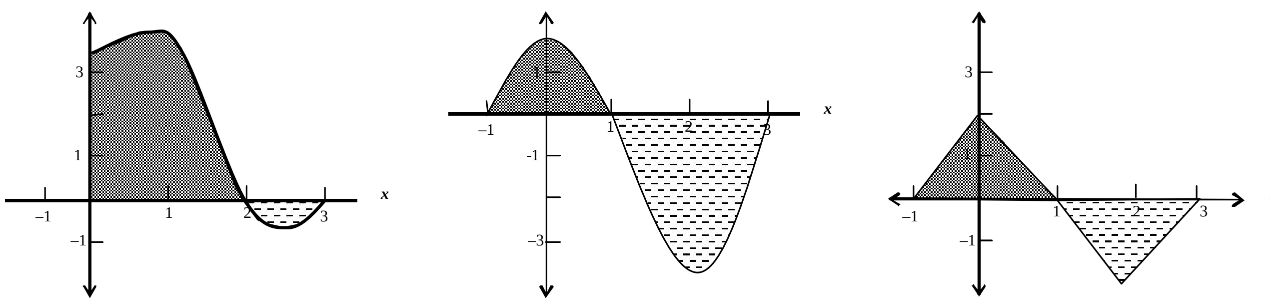
$$\int_1^3 (x^2-4x+3)\ dx = \frac{1}{3}x^3 - 2x^2 + 3x\Big]_1^3 = \left(\frac{1}{3}(3)^3 - 2(3)^2 + 3(3)\right) - \left(\frac{1}{3}(1)^3 - 2(1)^2 + 3(1)\right)$$

$$= (9-18+9) - \left(\frac{1}{3} - 2 + 3\right) = -\frac{4}{3}$$

This definite integral is negative because the shaded area lies below the $x$-axis. ◈

When a definite integral represents a portion of the graph that lies above as well as below the $x$-axis we can calculate two types of areas, **gross area** and **net area**. The gross area is the total amount of area that lies between the curve and the $x$-axis while the net area calculates how much more area lies above or below $x$-axis. Figure 9.7 shows the different values of the net area.

**Figure 9.??** Net Area.



Net area is positive because more area lies above the $x$-axis.

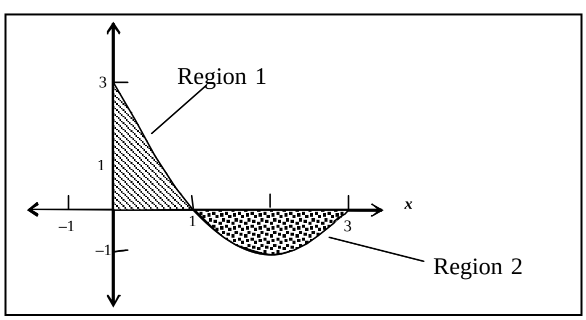
Net area is negative because more area lies below the $x$-axis

Net area is zero because the area above and below the $x$-axis is the same.

**Example 9.7** Draw a geometric representation of

$\int_0^3 (x^2-4x+3)\ dx$ and then calculate the net and gross areas.

**Solution** The shaded region in the graph below shows the geometric representation.



To find the gross area we need to evaluate the integral that represents each shaded region.

$$\text{Region}_1 = \int_0^1 (x^2 - 4x + 3)\, dx = \frac{1}{3}x^3 - 2x^2 + 3x\Big]_0^1 = \left(\frac{1}{3} - 2 + 3\right) - 0 = \frac{4}{3}$$

In Example 9.6, we found the area of region 2 to be $-4/3$. Therefore the gross area is

$$\text{Area of Region}_1 + \left|\text{Area of Region}_2\right| = \int_0^1 (x^2 - 4x + 3)\, dx + \left|\int_1^3 (x^2 - 4x + 3)\, dx\right|$$
$$= \frac{4}{3} + \left|-\frac{4}{3}\right| = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

The net area is just the sum of the two integrals,

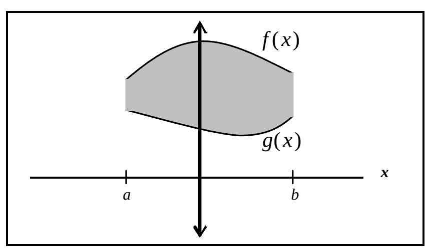
$$\int_0^3 (x^2 - 4x + 3)\, dx = \int_0^1 (x^2 - 4x + 3)\, dx + \int_1^3 (x^2 - 4x + 3)\, dx = \frac{4}{3} + \left(-\frac{4}{3}\right) = 0\,.$$

Since the net area is zero, we know there is the same amount of area above the *x*-axis as there is below the *x*-axis. Notice that calculating the function over the entire interval is another method of obtaining net area.

# 9.3 Area Between Two Curves

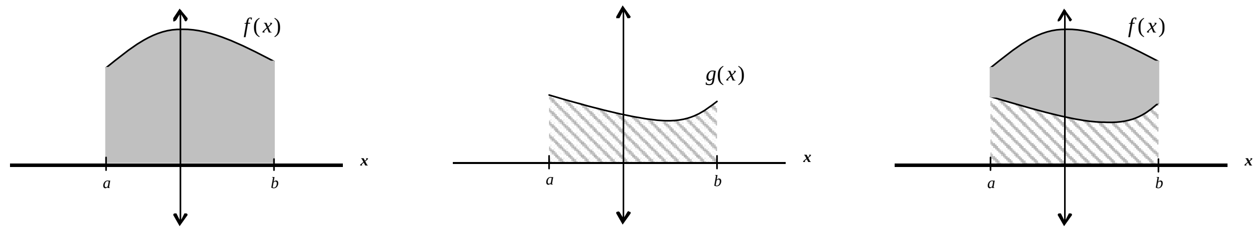
Suppose we are to find the area of the shaded region shown in Figure 9.??.

**Figure 9.??** The area between *f(x)* and *g(x)*.




The area under $f(x)$ on $[a, b]$ is shown in Figure 9.?? (a) and the area under $g(x)$ on $[a, b]$ is shown in Figure 9.?? (b). If the area under $g(x)$ is taken away from the area under $f(x)$ we obtain the area in Figure 9.?? (c) which is the area we were trying to find in Figure 9.??.

**Figure 9.??** Area Between Two Curves



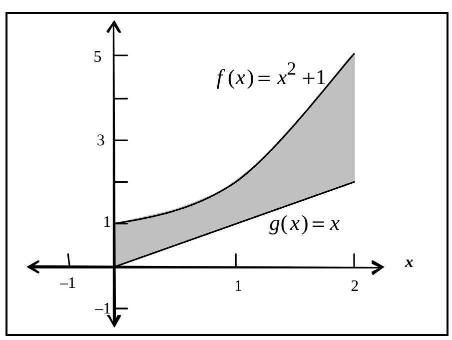

area under $f(x)$ on $[a, b]$ | area under $g(x)$ on $[a, b]$ | area under $f(x)$ with area under $g(x)$ taken away

Thus, we can find the area between two curves if we find the area under the top curve and subtract off the area under the bottom curve.

> Area Between Two Curves
>
> On the closed interval $[a, b]$, the area between two continuous functions *f(x)* and *g(x),* where $f(x) \geq g(x)$, is given by
>
> $$\int_a^b [f(x) - g(x)]\ dx$$
>
> The area between two curves can be remembered as $\int_a^b (\text{top function - bottom function})\,dx$

**Example 9.8** Find the area between $f(x) = x^2 + 1$ and $g(x) = x$ on [0, 2].

**Solution** First lets graph both functions over [0 2].

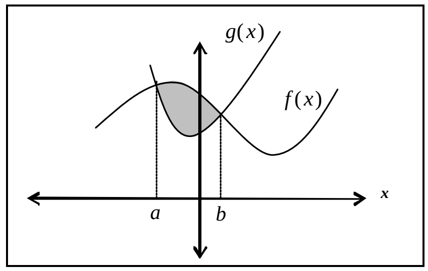



Since $f(x)$ is the top function and $g(x)$ is the bottom function, the definite integral, and thus the area between the two curves is

$$\int_0^2 \left[(x^2+1) - x\right]\ dx = \int_0^2 \left(x^2 - x + 1\right)\ dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + x\Big]_0^2 = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 + 2 - 0 = \frac{8}{3} - 2 + 2 = \frac{8}{3}$$

Sometimes the two given curves will intersect at one or more points, thus forming an area bounded by the curves as shown in Figure 9.??.

**Figure 9.??**

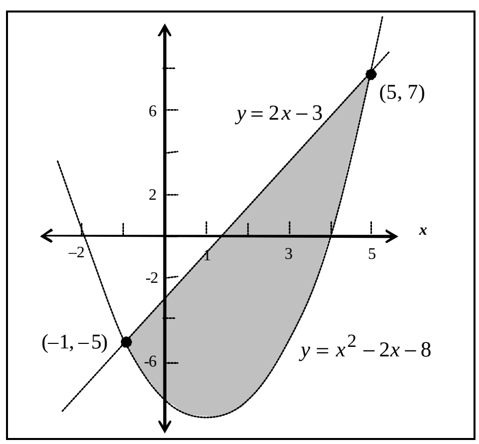



To find the area bounded by two curves we need to find the limits of integration. We do this by locating the points where the curves intersect. The definite integral for Figure 9.10 is represented by

$$\int_a^b [f(x) - g(x)]\,dx$$

**Example 9.9** Find the area of the region bounded by $y = x^2 - 2x - 8$ and $y = 2x - 8$.

**Solution** First, we need to graph the two functions on the same coordinate plane.




From the graph we notice that $y = 2x - 3$ is the top function and $y = x^2 - 2x - 8$ is the bottom function. In addition, the points of intersection show that the lower limit of integration is $x = -1$ and the upper limit of integration is $x = 5$. Thus, the definite integral is

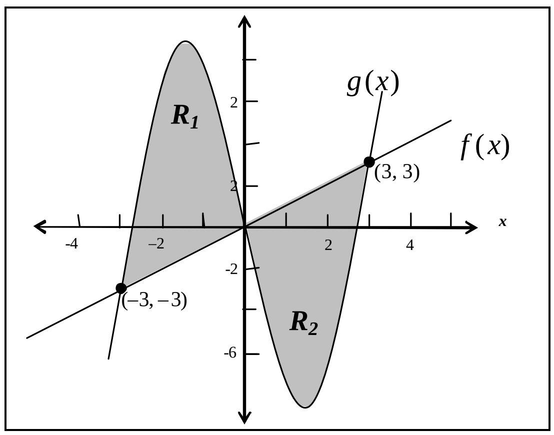
$$\int_{-1}^{5} (2x-3)-(x^2-2x-8)\ dx = \int_{-1}^{5} 2x-3-x^2+2x+8\ dx = \int_{-1}^{5} -x^2+4x+5\ dx$$

$$= -\frac{1}{3}x^3+2x^2+5x\Big]_{-1}^{5} = -\frac{1}{3}(125)+2(25)+5(5)-\left[-\frac{1}{3}(-1)+2(1)+5(-1)\right]$$

$$= \frac{100}{3}+\frac{8}{3} = \frac{108}{3} = 36$$

**Example 9.10** Find the area of the region bounded by $f(x) = x$ and $g(x) = x^3 - 8x$.

**Solution** First, we need to graph the two functions on the same coordinate plane.




There are two bounded regions ($R_1$ and $R_2$) produced by these curves. Notice that the top function of $R_1$ is *g(x)* and the top function of $R_2$ is *f(x)*. Consequently we will need to set up an integral to find the area of $R_1$, another integral to find the area of $R_2$, and then add the results.

$$R_1 = \int_{-3}^{0} (x^3 - 8x - x)\,dx = \int_{-3}^{0} (x^3-9x)\,dx = \frac{1}{4}x^4 - \frac{9}{2}x^2\Big]_{-3}^{0} = 0-\left(\frac{1}{4}(81)-\frac{9}{2}(9)\right) = \frac{81}{4} = 20.25$$

$$R_2 = \int_{0}^{3} (x^3 - 8x - x)\,dx = \int_{0}^{3} (x^3-9x)\,dx = \frac{1}{4}x^4 - \frac{9}{2}x^2\Big]_{0}^{3} = \frac{1}{4}(81)-\frac{9}{2}(9) = \frac{81}{4} = 20.25$$

Now adding the two results together we get

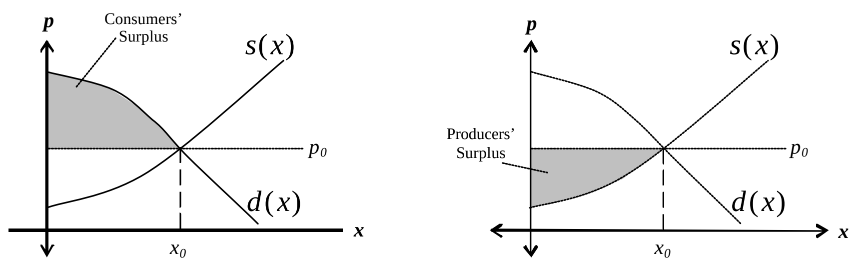
$$R_1 + R_2 = 20.25 + 20.25 = 40.5$$

# 9.4 Applications of Definite Integrals

## Consumers' and Producers' Surplus

Suppose you worked all summer and put away \$800 to buy a new stereo system for your dorm room. When you went shopping to buy the stereo system you found exactly what you wanted for only \$650. Thus, we could say that you "saved" \$150. If we could find all the consumers who were willing to pay over \$650 for this stereo system and calculate the total savings of all consumers, we will have found the **consumers' surplus.** Figure 9.11(a) shows the graph of a supply curve, $p = s(x)$, and a demand curve, $p = d(x)$. The dotted lines represents the equilibrium price, $p_0$, and the equilibrium quantity $x_0$,. The area above the dotted line, but below $d(x)$, would represent the consumers' surplus.

**Figure 9.??**




Now lets say you are the producer of the stereo systems and are willing to supply the stereos for \$500. If, however, you end up selling the stereos for \$650, you have "gained" \$150. The total amount gained over all possible prices is the **producers' surplus**. Figure 9.11 also shows the graph of the producers' surplus.

If $p = d(x)$ is the demand equation, $p = s(x)$ the supply equation, and $(x_0, p_0)$ is the equilibrium point then the consumers' surplus is given by

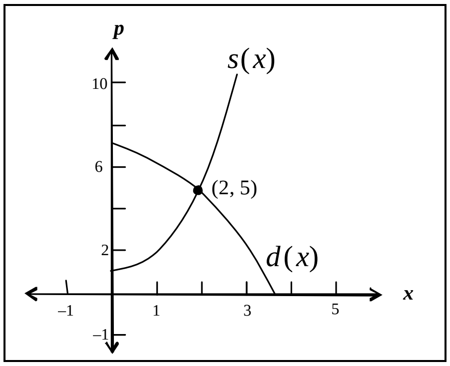
$$\int_0^{x_0} \left(d(x) - p_0\right) dx$$

the producers' surplus is given by

$$\int_0^{x_0} \left(p_0 - s(x)\right) dx$$

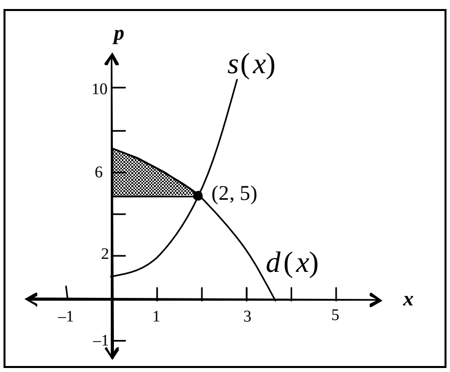
**Example 9.11** A company has determined that its supply and demand equations can be modeled by $p = d(x) = -\frac{1}{2}x^2 + 7$ and $p = s(x) = x^2 + 1$ where $x$ represents the number of units supplied each week and $p$ is the selling price (in hundreds of dollars) for each unit. Find the consumers' and producers' surplus.

**Solution** First we need to graph the supply and demand functions and find the equilibrium point. The equilibrium point is found by setting $d(x) = p(x)$.




$$\begin{aligned} d(x) &= s(x) \\ -\frac{1}{2}x^2 + 7 &= x^2 + 1 \\ 6 &= \frac{3}{2}x^2 \\ 12 &= 3x^2 \\ 4 &= x^2 \\ 2 &= x \end{aligned}$$

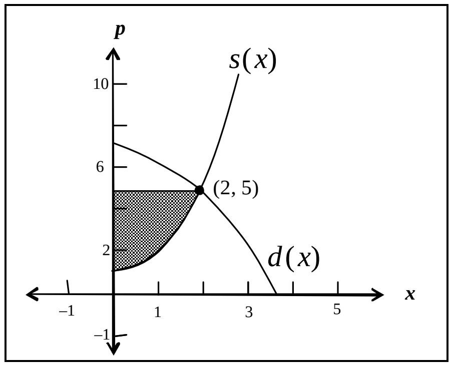
The consumers' surplus is




$$\int_0^2 \left(-\frac{1}{2}x^2 + 7 - 5\right)dx = \int_0^2 \left(-\frac{1}{2}x^2 + 2\right)dx = -\frac{1}{6}x^3 + 2x\Big]_0^2 = -\frac{1}{6}(8) + 2(2) = -\frac{4}{3} + 4 = \frac{8}{3}$$

So the consumers' "saved" approximately \$266.67 per week when the selling price was \$500.

The producers' surplus is



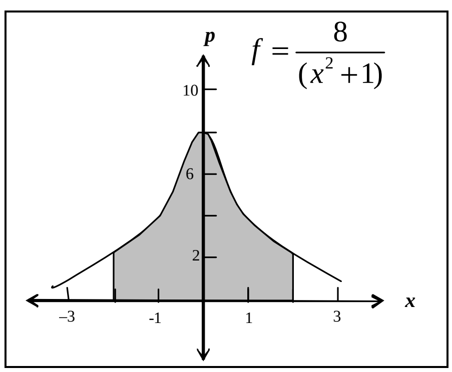

$$\int_0^2 \left(5-(x^2+1)\right)\,dx = \int_0^2 (-x^2+4)\,dx = -\frac{1}{3}x^3+4x\Big]_0^2 = -\frac{1}{3}(8)+4(2) = -\frac{8}{3}+8 = \frac{16}{3}$$

So the producers' "saved" approximately $533.33 per week when the selling price was $500. ◈

# Sample Quiz

**Question 9.1** Find an approximation for the area under $f(x) = -2x^2+2$ on $[-1,0]$ using 4 left endpoint rectangles and 4 right endpoint rectangles. Which is an overestimate and which is an underestimate?

**Question 9.2** Write a definite integral that represents the shaded area for the function graphed below.




**Question 9.3** Evaluate $\int_{-2}^{2} -x^2+4\ dx$.

**Question 9.4** Draw a graph of $f(x) = |x-2|$ and then find $\int_{-1}^{4} |x-2|\ dx$.

**Question 9.5** Evaluate $\int_{-1}^{2} x^3-x^2-4\ dx$.

**Question 9.6** Calculate the net and gross areas of $\int_1^8 x^2 - 10x + 21 \ dx$.

**Question 9.7** Find the area between $f(x) = 0.5x + 1$ and $g(x) = x^2 - 4x + 9$ on $[3, 6]$.

**Question 9.8** Find area bounded by $f(x) = x^2 + 4x + 2$ and $g(x) = 4x + 6$.

**Question 9.9** Find the area bounded by $f(x) = x^3 - 8.5x$ and $g(x) = 0.5x$.

**Question 9.10 A company has determined its demand equation can be modeled by** $p = d(x) = -0.25x^2 + 80$ and its supply equation can be modeled by $p = s(x) = 3.5x + 20$ where $x$ is the number of units sold per day and $p$ is the selling price in hundreds of dollars. Find the consumers' and producers' surplus.

# Chapter 10   Multi-Variable Applications

## Appendix A – Applets

## Appendix B – Trigonometry

## Answers to Sample Quizzes

## Answers to Odd Exercises

# Index