Effective Non-Vanishing of Class Group *L*-Functions for Biquadratic CM Fields

Emily Peirce (joint with Adrian Barquero-Sanchez and Katy Weber)

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Theorem (B-S, Peirce, W)

Let $d_1 > 0$ and $d_2 < 0$ be square-free, co-prime integers with $d_1 \equiv 1 \mod 4$ and $d_2 \equiv 2$ or $3 \mod 4$. Assume $K = \mathbb{Q}(\sqrt{d_1})$ has class number one and let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Then if

$$|d_2| \geq C_1(d_1) := (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1) + 2)\right\},$$

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then there exists a character $\chi \in \widehat{Cl(\mathcal{O}_E)}$ such that $L(\chi, \frac{1}{2}) \neq 0$.

Connection to Eisenstein Series

Under our assumptions on K and E, the average formula becomes

$$\frac{1}{h_E}\sum_{\chi\in\widehat{Cl(\mathcal{O}_E)}}L(\chi,\frac{1}{2})=\left(\frac{2^nd_1}{\sqrt{|d_2|}}\right)^{\frac{1}{2}}\frac{1}{[\mathcal{O}_E^{\times}:\mathcal{O}_K^{\times}]}E_K(z_{\mathcal{O}_E},\frac{1}{2})$$

where the special point is

$$z_{\mathcal{O}_E} = \left(\sqrt{d_2}, \sqrt{d_2}\right) \in \mathbb{H}^2.$$

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where the special point is

$$z_{\mathcal{O}_E} = \left(\sqrt{d_2}, \sqrt{d_2}\right) \in \mathbb{H}^2.$$

From this formula, it suffices to show that

$$E_{\mathcal{K}}(z_{\mathcal{O}_{\mathcal{E}}},\frac{1}{2})\neq 0.$$

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Decomposition of the Eisenstein Series

Proposition

We have

$$E_{\mathcal{K}}(z_{\mathcal{O}_{E}},\frac{1}{2}) = M(d_{1},d_{2}) + H(d_{1},d_{2})$$

where

$$M(d_1, d_2) = \sqrt{|d_2|} \left[\frac{2R_{\mathcal{K}}}{\sqrt{d_1}} \left(\log\left(|d_2|\right) - \log\left(\frac{\pi^2}{d_1}\right) - 2(\gamma_{\mathbb{Q}} + \log(4)) \right) + 2\gamma_{\mathcal{K}} \right]$$

and

$$H(d_1, d_2) = \sqrt{|d_2|} \sum_{\gamma \in \mathcal{O}_K} \sum_{0 \neq \nu \in \mathcal{O}_K^{\vee}} c_{\nu}(\gamma y(z_{\mathcal{O}_E})) e^{2\pi i Tr(\gamma \nu x)}$$

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• By the previous proposition and the reverse triangle inequality,

$$|E_{\mathcal{K}}(z_{\mathcal{O}_{d_2}}, \frac{1}{2})| \geq |M(d_1, d_2)| - |H(d_1, d_2)|.$$

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• Hence it suffices to show $|M(d_1, d_2)| > |H(d_1, d_2)|$.

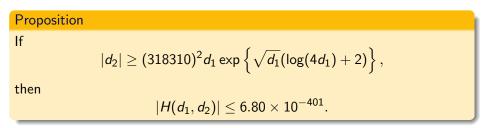
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$$|E_{\mathcal{K}}(z_{\mathcal{O}_{d_2}}, \frac{1}{2})| \geq |M(d_1, d_2)| - |H(d_1, d_2)|.$$

- Hence it suffices to show $|M(d_1, d_2)| > |H(d_1, d_2)|$.
- We will give an upper bound for $|H(d_1, d_2)|$ and a lower bound for $|M(d_1, d_2)|$.

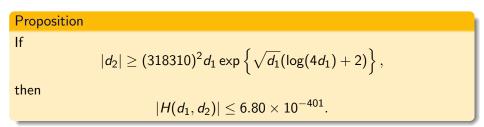
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An Upper Bound for $|H(d_1, d_2)|$





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The proof involves a very complicated argument to effectivize an upper bound of Bauer.

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A Lower Bound for $M(d_1, d_2)$

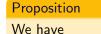
Proposition

We have

 $M(d_1, d_2) > 1.$



A Lower Bound for $M(d_1, d_2)$



$$M(d_1,d_2)>1.$$

The proof uses Ihara's lower bound

$$\gamma_{\mathcal{K}} > -2(\log(4d_1)+2)(\log(\sqrt{d_1})-\gamma_{\mathbb{Q}}-\log(4\pi))$$

and the lower bound

 $R_{\mathcal{K}} > \log(2\sqrt{d_1}).$

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Summary

• For $|d_2| \ge (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1)+2) ight\},$ we have $|H(d_1,d_2)| < 1$ and $M(d_1,d_2) > 1.$

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- Thus $|M(d_1, d_2)| > |H(d_1, d_2)|$, implying $|E_K(z_{\mathcal{O}_E}, \frac{1}{2})| > 0$.
- Hence, by the average formula, there exists a $\chi \in \widehat{Cl(\mathcal{O}_E)}$ such that $L(\chi, \frac{1}{2}) \neq 0$.

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• Hence for all

 $|d_2| \ge 2.77028 \times 10^{13},$

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there exists a χ such that $L(\chi, \frac{1}{2}) \neq 0$.

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- A version of the main result holds for *any* CM extension *E* of *K* when *K* has class number one.

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• One has reduced the question of the existence of non-vanishing $L(\chi, \frac{1}{2})$ to a (large!) finite calculation.