A problem coming from group theory

Suho Oh

Texas State University

May 7, 2016
Original Problem

For each $i, j$, a set $B_{i,j}$ is an arbitrary set of cardinality $n$.

$a_{i,1} \cdots a_{i,n-1} a_{i,n} \cdots$
$a_{2,1} \cdots a_{2,n-1} a_{2,n} \cdots$
$: \cdots \cdots \cdots \cdots \cdots$
$a_{n,1} \cdots \cdots a_{n,n} \cdots$

Easy when all $B_{i,j} = [n]$. Try!

A problem coming from group theory

May 7, 2016
For each $i, j$, a set $B_{i,j}$ is an arbitrary set of cardinality $n$.

$\mathbf{a}_{i,j} \in B_{i,j}$. 
For each $i, j$, a set $B_{i,j}$ is an arbitrary set of cardinality $n$.

- $a_{i,j} \in B_{i,j}$.
- (first $n-1$ columns) $a_{i,1}, \ldots, a_{i,n-1}$ are mutually distinct.
For each $i, j$, a set $B_{i,j}$ is an arbitrary set of cardinality $n$.

For each $i, j$, $a_{i,j} \in B_{i,j}$.

(first n-1 columns) $a_{i,1}, \ldots, a_{i,n-1}$ are mutually distinct.

(afterwards) $a_{i,t} \notin \{a_{i,1}, \ldots, a_{i,n-2}\}$.
For each $i, j$, a set $B_{i,j}$ is an arbitrary set of cardinality $n$.

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(afterwards) $a_{i,t} \not\in \{a_{i,1}, \cdots, a_{i,n-2}\}$.

(row sets distinct at any point) $\{a_{i,1}, \cdots, a_{i,k}\} \neq \{a_{j,1}, \cdots, a_{j,k}\}$

\[
\begin{array}{cccc}
  a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\
  a_{2,1} & \cdots & a_{2,n-1} & a_{2,n} \\
  \vdots & \cdots & \cdots & \cdots \\
  a_{n,1} & \cdots & \cdots & a_{n,n} \\
\end{array}
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For each $i, j$, a set $B_{i,j}$ is an arbitrary set of cardinality $n$.

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Easy when all $B_{i,j} = [n]$. Try!
Example

When $n = 5$, all $B_{i,j} = \{1, 2, 3, 4, 5\}$

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When all $B_{i,j}$ are the same, easy!

Observe: First $n-1$ columns and the columns after behave slightly differently.

If all $B_{i,j}$ are different, also easy!

Harder for other cases!

Number of columns? Can we just think of finite cases?

Konig Lemma: For a connected graph with infinitely many vertices, where degree is finite, the graph contains an infinitely long simple path.

(TSU)
Example

When \( n = 5 \), all \( B_{i,j} = \{1, 2, 3, 4, 5\} \)

\[
\begin{align*}
1 & \ 2 \ 3 \ 4 \ 4 \ 4 \ \cdots \\
2 & \ 3 \ 4 \ 5 \ 5 \ 5 \ \cdots \\
3 & \ 4 \ 5 \ 1 \ 1 \ 1 \ \cdots \\
4 & \ 5 \ 1 \ 2 \ 2 \ 2 \ \cdots \\
5 & \ 1 \ 2 \ 3 \ 3 \ 3 \ \cdots 
\end{align*}
\]

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2 & 3 & 4 & 5 & 5 & 5 & \ldots \\
3 & 4 & 5 & 1 & 1 & 1 & \ldots \\
4 & 5 & 1 & 2 & 2 & 2 & \ldots \\
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- Harder for other cases!
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- Konig Lemma: \( G \) a connected graph with infinitely many vertices, where degree is finite, \( G \) contains an infinitely long simple path.
Motivation

\[
\begin{array}{cccc}
  a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\
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  a_{n,1} & \cdots & \cdots & a_{n,n} \\
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Motivation

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$G$ group acting on $G$-module $V$. 

(Conjecture by Moreto, Jaikin-Zapairin) $r_k(G)$ is bounded linearly by $n(G, V)$. 

$dl(G)$: derived length of a solvable group. (Keller) $dl(G) \leq 24 \log n(G, V) + 364$ (Conjectured by Keller) $dl(G) \leq 6 \log n(G, V) + 6$ (Follows from the problem for $n=6$) (Curtin O.) Yup! Problem is true for any $n$. 

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**Motivation**

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- $G$ group acting on $G$-module $V$.
- $n(G, V)$ : number of orbit sizes of $G$ on $V$. 

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Yup! Problem is true for any $n$. (TSU: A problem coming from group theory)
Motivation

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a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} & \cdots \\
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\vdots & \cdots & \cdots & \cdots & \cdots \\
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\end{array} \]

- \( G \) group acting on \( G \)-module \( V \).
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\[ a_{1,1} \quad \cdots \quad a_{1,n-1} \quad a_{1,n} \quad \cdots \]
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& \quad \vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \\
& a_{n,1} \quad \cdots \quad \cdots \quad a_{n,n} \quad \cdots 
\end{align*}

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Generalization

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\begin{array}{cccccc}
  a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} & \cdots \\
  \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  a_{n,1} & \cdots & \cdots & a_{n,n} & \cdots \\
\end{array}
\]

\[\text{(*, afterwards)}\]

\[a_{i,t} \not\in \{a_{i,1}, \cdots, a_{i,k} \} \text{ for row sets distinct at any point}\]

\[\{a_{i,1}, \cdots, a_{i,k}\} \neq \{a_{j,1}, \cdots, a_{j,k}\}\]

Observation:

Each \(a_{1,n-1}, a_{1,n}, a_{1,n+1}, \cdots\) have at least two options to choose from, to obey (*).

Those two options may have already appeared in the row set \(\{a_{i,1}, \cdots, a_{i,k}\}\). In terms of row sets, only consider when both are new.

Ex: 1234 | 567

\(B = \{1, 2, 3, 4, 5, 8\}\)

We choose 5, row set is same \(\{1, 2, 3, 4, 5, 6, 7\}\).

Ex: 1234 | 567

\(B = \{1, 2, 3, 4, 8, 9\}\)

We choose 9, row set is now \(\{1, 2, 3, 4, 5, 6, 7, 9\}\).
Generalization

\[
\begin{array}{cccccc}
  a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} & \cdots \\
  \vdots & \cdots & \cdots & \cdots & \cdots \\
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\end{array}
\]

(first n-1 columns) \( a_{i,1}, \cdots , a_{i,n-1} \) are mutually distinct.

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Those two options may have already appeared in the row set \(\{a_{i,1}, \cdots, a_{i,k}\}\). In terms of row sets, only consider when both are new.
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Growth of $n$-sets, where you are offered two options each time!
System of distinct representative of $n$ binary posets

- Growth of $n$-sets, where you are offered two options each time!
- Binary poset: Each node has two children below.

Boolean lattice $B_n$, $n$-binary posets rooted at 1, 2, 3, ···, $n$.

Can you find a chain for each binary poset so that the chains are pairwise disjoint?

$$(n-1)$$-partite graph. Hypergraph where edges are chains along the binary posets.

Can you find a system of distinct representative for the given $n$ hypergraphs?

(Aharoni) Hall's theorem for hypergraphs... Not easy to use.
System of distinct representative of $n$ binary posets

- Growth of $n$-sets, where you are offered two options each time!
- Binary poset: Each node has two children below.
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Binary poset diagram: 

```
1 2 3 4
12 13 14 23 24 34
123 124 134 234
```
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Adversary game version of the problem

If you can come up with a strategy that only requires you to look at one level at a time..

You start with \( n \) sets \( \{A_1, A_2, \ldots, A_n\} \), where \( A_1 = \{1\}, A_2 = \{2\}, \ldots, A_n = \{n\} \). Each turn, for each set \( A_i \), adversary chooses 2 elements in \([n] \setminus A_i\). For each set, you must pick the correct element among the 2 choices so that all \( A_i \)'s are still mutually distinct!

Ex:

\( A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, A_4 = \{4\} \).

Adversary offers \( \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 2\} \) each.

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Done!

Can't be too crowded! Ex: 12345, 12346, 12347, 12356, 12367, offered 67, 57, 56, 47, 45.
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Ex: $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, $A_4 = \{4\}$. Adversary offers $\{2, \ 3\}$, $\{3, \ 4\}$, $\{4, \ 1\}$, $\{1, \ 2\}$ each.

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(TSU) A problem coming from group theory
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Additional comments
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- Generalized Hall: Given hypergraphs $H_1, \cdots, H_n$, when can you find $e_i \in H_i$'s such that they are pairwise disjoint?

(Clean but is only sufficient): For any $I \subseteq [n]$, there is a matching (set of disjoint edges) $M_I$ in $G_I = \bigcup I H_I$ which cannot be pinned (Not all edges of $M_I$ are touched) by fewer than $|I|$ disjoint edges in $G_I$.

Sadly, this doesn't work even for cases like when all $B$'s are the same!!

(Sufficient and necessary): For each $I \subseteq [n]$, you can find a matching $M_I$ in $G_I$ such that $M_I$ cannot be pinned by fewer than $|I|$ edges from $\bigcup J \subseteq I M_J$.

For the adversary problem: we want to prevent the sets being too crowded. Try maintaining distance each step!

Required condition: For each $I \subseteq [n]$, let $A_{i_1}, \cdots, A_{i_k}$ be the sets containing $I$ in our collection. Then $|A_{i_1} \cup \cdots \cup A_{i_k}| \geq k + |I|$.

But still not enough..
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