# Odd behavior in the coefficients of reciprocals of binary power series

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# Introduction

Recall that every number has a unique binary representation and can be written as $\sum_{j=0}^{\infty} c_j 2^j$, where $c_j \in \{0, 1\}$.

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Recall that every number has a unique binary representation and can be written as $\sum_{j=0}^{\infty} c_j 2^j$, where $c_j \in \{0, 1\}$.

Question: What happens if we take the coefficients from a different set?

## The Stern Sequence

Example: If we take coefficients from the set $\{0, 1, 2\}$, then the binary representation is no longer unique. For example, there are three ways to write $n = 4$ as $\sum \epsilon_i 2^i,\ \epsilon_i \in \{0, 1, 2\}$:

$$4 = 2 \cdot 1 + 1 \cdot 2 = 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 2^2 = 0 \cdot 1 + 2 \cdot 2.$$

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$$4 = 2 \cdot 1 + 1 \cdot 2 = 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 2^2 = 0 \cdot 1 + 2 \cdot 2.$$

Taking coefficients from this set, the number of representations of $n - 1$ corresponds to the $n$th term in the Stern sequence, which is defined by $s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$, with $s(0) = 0$ and $s(1) = 1$.

1 1

1 2 1

1 3 2 3 1

1 4 3 5 2 5 3 4 1

1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 1

1 6 5 9 4 11 7 10 3 11 8 13 5 12 7 9 2 9 7 12 5 13 8 11 3 10 7 11 4 9 5 6 1

. . . 14 11 19 8 21 13 18 5 17 12 19 7 16 9 11 2 11 9 16 7 19 12 17 5 18 13 21 8 19 11 14 . . .

# Generalizing the Ideas

Let $\mathcal{A} = \{0 = a_0 < a_1 < \cdots < a_j\}$ denote a finite subset of $\mathbb{N}$ containing 0. Let $f_{\mathcal{A}}(n)$ denote the number of ways to write $n$ in the form

$$n = \sum_{k=0}^{\infty} \epsilon_k 2^k, \quad \epsilon_k \in \mathcal{A}.$$

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$$n = \sum_{k=0}^{\infty} \epsilon_k 2^k, \quad \epsilon_k \in \mathcal{A}.$$

We associate to $\mathcal{A}$ its characteristic function $\chi_{\mathcal{A}}(n)$ and the generating function

$$\phi_{\mathcal{A}}(x) := \sum_{n=0}^{\infty} \chi_{\mathcal{A}}(n) x^n = \sum_{a \in \mathcal{A}} x^a = 1 + x^{a_1} + \cdots + x^{a_j}.$$

# Product Representation

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Viewing the number of ways to write $n$ as a partition problem, we obtain the following product representation for $F_{\mathcal{A}}(x)$.

$$F_{\mathcal{A}}(x) = \prod_{k=0}^{\infty} \left(1 + x^{a_1 2^k} + \cdots + x^{a_j 2^k}\right) = \prod_{k=0}^{\infty} \phi_{\mathcal{A}}(x^{2^k})$$

In *Congruence Properties of Binary Partition Functions*, Anders, Dennsion, Lansing, and Reznick studied the behavior of $(f_{\mathcal{A}}(n)) \bmod 2$. Theorem 1.1 states that

$$\phi_{\mathcal{A}}(x)F_{\mathcal{A}}(x) = 1 \quad \text{in } \mathbb{F}_2[x].$$

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$$\phi_{\mathcal{A}}(x)F_{\mathcal{A}}(x) = 1 \quad \text{in } \mathbb{F}_2[x].$$

We can make similar definitions for an infinite set $\mathcal{A}$ containing 0, and the above result still applies. This relates our work to work by Cooper, Eichhorn, and O'Bryant.

# Return to the Stern Sequence

| $n$ | $f_{\{0,1,2\}}(n)$ | $s(n)$ | $n$ | $f_{\{0,1,2\}}(n)$ | $s(n)$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 9 | 3 | 4 |
| 1 | 1 | 1 | 10 | 5 | 3 |
| 2 | 2 | 1 | 11 | 2 | 5 |
| 3 | 1 | 2 | 12 | 5 | 2 |
| 4 | 3 | 1 | 13 | 3 | 5 |
| 5 | 2 | 3 | 14 | 4 | 3 |
| 6 | 3 | 2 | 15 | 1 | 4 |
| 7 | 1 | 3 | 16 | 5 | 1 |
| 8 | 4 | 1 | | | |

Stern noticed in 1858 that the parity of $s(n)$ is periodic with period 3, and Reznick proved in 1989 that $s(n) = f_{\{0,1,2\}}(n-1)$.

# Example

Note that $\phi_{\{0,1,2\}}(x) = 1 + x + x^2$, and applying the theorem, we see that in $F_2[[x]]$,

$$\begin{aligned} F_{\{0,1,2\}}(x) &= \frac{1}{1+x+x^2} \\ &= \frac{1+x}{1+x^3} \\ &= (1+x)(1+x^3+x^6+\cdots) \\ &= 1+x+x^3+x^4+x^6+x^7+\cdots \end{aligned}$$

# Another Example

Dennison observed in her thesis that if $\mathcal{A} = \{0, 1, 3\}, f_{\mathcal{A}}(n)$ is periodic with period 7 and each period has four odd terms. Specifically, $f_{\mathcal{A}}(n)$ is odd when $n \equiv 0, 1, 2, 4 \pmod{7}$.

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Using our main theorem, we find that in $F_2[[x]]$,

$$F_{\{0,1,3\}}(x) = \frac{1}{1 + x + x^3} = \frac{1 + x + x^2 + x^4}{1 + x^7}.$$

Similarly, Dennison noted that if $\mathcal{A} = \{0, 2, 3\}$, $f_{\mathcal{A}}(n)$ is periodic with period 7 and each period has four odd terms, which occur when $n \equiv 0, 2, 3, 4 \pmod 7$.

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Again, it follows from our main theorem that

$$F_{\{0,2,3\}}(x) = \frac{1}{1 + x^2 + x^3} = \frac{1 + x^2 + x^3 + x^4}{1 + x^7}.$$

# Definitions and Observations

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- Define $p^*(x)$ by $p(x)p^*(x) = 1 + x^D$.

# An Example from Our Paper

Let $\mathcal{A} = \{0, 1, 4, 9\}$. In $\mathbb{F}_2[x]$,

$$\phi_{\mathcal{A}} = 1 + x + x^4 + x^9 = (1 + x)^4(1 + x + x^2)(1 + x^2 + x^3).$$

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Quick computations show that the period of $\phi_{\mathcal{A}}$ is 84. Recall that this means $\phi_{\mathcal{A}}\phi^*_{\mathcal{A}} = 1 + x^{84}$. Further computations show that $\phi^*_{\mathcal{A}}$ has 41 terms with exponents in the set $\{0, 1, 2, 3, \ldots, 70, 75\}$.

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As we will see, this means that $\left(f_{\{0,1,4,9\}}(n) \bmod 2\right)$ is periodic with period 84 and has 41 odd terms and 43 even terms in each period.

We have

$$F_{\mathcal{A}}(x) = \frac{1}{\phi_{\mathcal{A}}(x)} = \frac{\phi^*_{\mathcal{A}}(x)}{1 + x^D} \quad \text{in } \mathbb{F}_2[x]. \tag{1}$$

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If $\phi^*_{\mathcal{A}}(x) = \sum_{i=1}^{r} x^{b_i}$, where $0 = b_1 < \cdots < b_r = D - \max \mathcal{A}$, then

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In any block of $D$ consecutive integers,

$$\#\{n : f_{\mathcal{A}}(n) \text{ is odd}\} = \ell(\phi^*_{\mathcal{A}}) = \beta_1(\phi_{\mathcal{A}})$$
$$\#\{n : f_{\mathcal{A}}(n) \text{ is even}\} = D - \ell(\phi^*_{\mathcal{A}}) = \beta_0(\phi_{\mathcal{A}}).$$

In *Reciprocals of Binary Power Series*, which appeared in *International Journal of Number Theory* in 2006, Cooper, Eichhorn, and O'Bryant considered the fraction $\ell(\phi^*_{\mathcal{A}})/D$, as we did in our paper. Here I instead consider the ordered pair

$$\beta(\phi_{\mathcal{A}}) := (\beta_1(\phi_{\mathcal{A}}), \beta_0(\phi_{\mathcal{A}})),$$

which gives more detailed information than reduced fractions.

The first coordinate represents the number of times $f_{\mathcal{A}}(n)$ is odd in a minimal period, and the second coordinate represents the number of times $f_{\mathcal{A}}(n)$ is even in a minimal period.

# Robust polynomials

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We call a polynomial $f(x)$ *robust* if $\beta_1(f) > \beta_0(f) + 1$. This is equivalent to saying that $\beta_1(f) > (D+1)/2$, where $D$ is the order of $f(x)$.

They also posed the problem of describing the set

$$\left\{ \frac{\beta_1(f)}{\beta_0(f) + \beta_1(f)} : f(x) \text{ is a polynomial} \right\}.$$

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Since $f(x) = 1 + x^D$ has order D and $\beta_1(f) = \ell(f^*(x)) = 1$, we see the greatest lower bound of the set is 0. I will exhibit four sequences $\{f_n\}$ of polynomials such that $\beta_1(f_n) - \beta_0(f_n) \to \infty$, and, moreover,

$$\lim_{n\to\infty} \frac{\beta_1(f_n)}{\beta_0(f_n) + \beta_1(f_n)} = 1.$$

For $n$ with standard binary representation

$$n = 2^{b_k} + 2^{b_{k-1}} + \cdots + 2^{b_1} + 2^{b_0},$$

define

$$P_n(x) = x^{b_k} + x^{b_{k-1}} + \cdots + x^{b_1} + x^{b_0}.$$

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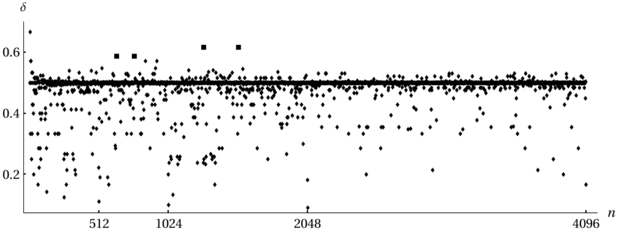
For example, $11 = 2^3 + 2^1 + 2^0$, so $P_{11}(x) = x^3 + x + 1$. For odd $n$, consider the fraction

$$\frac{\ell\left(P_n^*\right)}{\mathrm{ord}(P_n)}.$$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# Reciprocal Polynomials

## Definition

For a polynomial $f(x)$ of degree $n$, the *reciprocal polynomial* of $f(x)$ is $f_{(R)}(x) := x^n f(1/x)$.

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If $\text{order}(f(x)) = D$, then $\text{order}\left(f_{(R)}(x)\right) = D$. Thus $\beta(f(x)) = \beta\left(f_{(R)}(x)\right)$, and the robustness of $f(x)$ is equivalent to the robustness of $f_{(R)}(x)$.

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With $\mathcal{A} = \{0 = a_0 < a_1 < \cdots < a_j\}$, define

$$\tilde{\mathcal{A}} = \{0, a_j - a_{j-1}, \cdots, a_j - a_1, a_j\}.$$

Then $\phi_{\mathcal{A},(R)}(x) = \phi_{\tilde{\mathcal{A}}}$.

# First Theorem

## Theorem

*Fix $r \geq 3$.*

(i) *The order of* $f_{r,1}(x) := (1+x)(1+x^{2^r-1}+x^{2^r})$ *divides* $4^r - 1$.

(ii) $\beta_1(f_{r,1}) = 4^r - 3^r$

(iii) *Hence* $\beta(f_{r,1}) = (4^r - 3^r, 3^r - 1)$ *and* $f_{r,1}(x)$ *is robust.*

# Example

Consider $f_{3,1}(x) = 1 + x + x^7 + x^9$.

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- $\text{order}\,(f_{3,1}(x)) = 4^3 - 1 = 63$
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- $\beta\,(f_{3,1}) = (37, 26)$

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# Proof

Define

$$g_{r,1}(x) = \prod_{j=0}^{r-1} \left(1 + x^{(2^r-1)2^j} + x^{2^r 2^j}\right) + x^{4^r-2^r}.$$

By a lemma,

$$\left(1 + x^{2^r-1} + x^{2^r}\right) g_{r,1}(x) = 1 + x^{4^r-1}.$$

Because

$$g_{r,1}(1) = \prod_{j=0}^{r-1} (1 + 1 + 1) + 1 \equiv 0 \pmod 2,$$

we know $(1 + x) \mid g_{r,1}(x)$.

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Write $(1 + x)h_{r,1}(x) = g_{r,1}(x)$, so

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Thus $f_{r,1}(x) \mid \left(1 + x^{4^r-1}\right)$ and

$$f_{r,1}h_{r,1} = 1 + x^{4^r - 1}.$$

Rewrite

$$g_{r,1}(x) = \prod_{j=0}^{r-1} \left(1 + x^{(2^r-1)2^j} + x^{2^r 2^j}\right) + x^{4^r - 2^r}$$

to obtain

$$g_{r,1}(x) = \prod_{j=0}^{r-1} \left(1 + x^{(2^r-1)2^j}(1 + x^{2^j})\right) + x^{4^r - 2^r}.$$

Expand the product and rewrite, using $1+x^{2^j}=(1+x)^{2^j}$, to obtain

$$\begin{aligned} g_{r,1}(x) &= 1+x^{4^r-2^r}+\sum_{n=1}^{2^r-1} x^{(2^r-1)n}(1+x)^n \\ &= (1+x)\left(\frac{1+x^{4^r-2^r}}{1+x}+\sum_{n=1}^{2^r-1} x^{(2^r-1)n}(1+x)^{n-1}\right) \\ &= (1+x)\left(\sum_{j=0}^{4^r-2^r-1} x^j+\sum_{n=1}^{2^r-1} x^{(2^r-1)n}(1+x)^{n-1}\right). \end{aligned}$$

Ultimately, $(\beta_1(f_{r,1}), \beta_0(f_{r,1})) = (4^r-3^r, 3^r-1)$.

## Corollary

*The reciprocal polynomials* $f_{(R),r,1} = (1+x)(1+x+x^{2^r})$ *are also robust with order dividing* $4^r - 1$.

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Consider $f_{(R),3,1}(x) = 1 + x^2 + x^8 + x^9$.

- order $f_{(R),3,1} = 4^3 - 1 = 63$
- $\beta\left(f_{(R),3,1}\right) = (37, 26)$

### Theorem

*Fix* $r \geq 3$.

(i) *The order of* $f_{r,2}(x) := (1+x)(1+x^{2^r}+x^{2^r+1})$ *divides* $4^r + 2^r + 1$.

(ii) $\beta_1(f_{r,2}) = 4^r - 3^r + 2^r$

(iii) $\beta(f_{r,2}) = (4^r - 3^r + 2^r, 3^r + 1)$ *and* $f_{r,2}(x)$ *is robust.*

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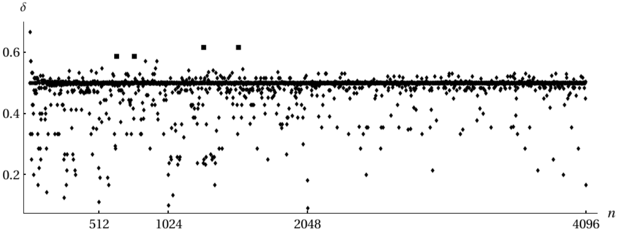
### Example

Consider $f_{(R),3,2}(x) = 1 + x^2 + x^9 + x^{10}$.

- order $f_{(R),3,2} = 73$
- $\beta\left(f_{(R),3,2}\right) = (45, 28)$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# Future Research Ideas

- Finding more families of robust polynomials
- Determining the cluster points of

$$\left\{ \frac{\beta_1(f)}{\beta_0(f)+\beta_1(f)} : f(x) \text{ is a polynomial} \right\}$$

- Exploring properties of $f_{\mathcal{A}}(n)$ in bases other than 2

# Acknowledgements


- The presenter acknowledges support from National Science Foundation grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students".
- The presenter also wishes to thank Professor Bruce Reznick for his time, ideas, and encouragement.

Recall $f_{\mathcal{A}}(n)$ is the number of ways to write

$$n = \sum_{i=0}^{\infty} \epsilon_i 2^i, \text{ where } \epsilon_i \in \mathcal{A} := \{0 = a_0 < a_1 < \cdots < a_z\}.$$

Expanding the sum, we see that

$$\begin{aligned} n &= \epsilon_0 + \epsilon_1 2 + \epsilon_2 2^2 + \cdots \\ &= \epsilon_0 + 2\left(\epsilon_1 + \epsilon_2 2 + \cdots\right) \end{aligned}$$

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We will now examine the asymptotic behavior of

$$\sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n).$$

Write $\mathcal{A} = \{0 = 2b_1, 2b_2, \ldots, 2b_s, 2c_1 + 1, \ldots, 2c_t + 1\}$.

Write $\mathcal{A} = \{0 = 2b_1, 2b_2, \ldots, 2b_s, 2c_1 + 1, \ldots, 2c_t + 1\}$.

If $n$ is even, then $\epsilon_0 = 0, 2b_2, 2b_3, \ldots$, or $2b_s$ and

$$f_{\mathcal{A}}(n) = f_{\mathcal{A}}\left(\frac{n}{2}\right) + f_{\mathcal{A}}\left(\frac{n - 2b_2}{2}\right) + f_{\mathcal{A}}\left(\frac{n - 2b_3}{2}\right) + \cdots + f_{\mathcal{A}}\left(\frac{n - 2b_s}{2}\right).$$

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Writing $n = 2\ell$, we have

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - b_2) + f_{\mathcal{A}}(\ell - b_3) + \cdots + f_{\mathcal{A}}(\ell - b_s).$$

If $n$ is odd, then $\epsilon_0 = 2c_1 + 1, 2c_2 + 1, \ldots,$ or $2c_t + 1$, and

$$f_{\mathcal{A}}(n) = f_{\mathcal{A}}\left(\frac{n-(2c_1+1)}{2}\right) + f_{\mathcal{A}}\left(\frac{n-(2c_2+1)}{2}\right) + \cdots + f_{\mathcal{A}}\left(\frac{n-(2c_t+1)}{2}\right).$$

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Writing $n = 2\ell + 1$, we have

$$f_{\mathcal{A}}(2\ell+1) = f_{\mathcal{A}}(\ell - c_1) + f_{\mathcal{A}}(\ell - c_2) + \cdots + f_{\mathcal{A}}(\ell - c_t).$$

# Example

If $\mathcal{A} = \{0, 1, 4, 9\} = \{2 \cdot 0, 2 \cdot 0 + 1, 2 \cdot 2, 2 \cdot 4 + 1\}$, then we have

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 2)$$

and

$$f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 4).$$

For positive integers $k, m$, and $a_z$, let

$$\omega_k(m) = \begin{pmatrix} f_{\mathcal{A}}(2^k m) \\ f_{\mathcal{A}}(2^k m - 1) \\ \vdots \\ f_{\mathcal{A}}(2^k m - a_z) \end{pmatrix}.$$

We will show that for $a_z$ sufficiently large, there exists a fixed $(a_z + 1) \times (a_z + 1)$ matrix $M$ such that for any $k \geq 0$,

$$\omega_{k+1} = M\omega_k.$$

# Example

Let $\mathcal{A} = \{0, 1, 3, 4\}$. Then

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 2)$$

and

$$f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 1).$$

# $\{0, 1, 3, 4\}$ continued

$$\omega_{k+1}(m) = \begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m-1) \\ f_{\mathcal{A}}(2^{k+1}m-2) \\ f_{\mathcal{A}}(2^{k+1}m-3) \\ f_{\mathcal{A}}(2^{k+1}m-4) \end{pmatrix} = \begin{pmatrix} f_{\mathcal{A}}(2^{k}m) + f_{\mathcal{A}}(2^{k}m-2) \\ f_{\mathcal{A}}(2^{k}m-1) + f_{\mathcal{A}}(2^{k}m-2) \\ f_{\mathcal{A}}(2^{k}m-1) + f_{\mathcal{A}}(2^{k}m-3) \\ f_{\mathcal{A}}(2^{k}m-2) + f_{\mathcal{A}}(2^{k}m-3) \\ f_{\mathcal{A}}(2^{k}m-2) + f_{\mathcal{A}}(2^{k}m-4) \end{pmatrix}.$$

## $\{0, 1, 3, 4\}$ continued

$$\omega_{k+1}(m) = \begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m-1) \\ f_{\mathcal{A}}(2^{k+1}m-2) \\ f_{\mathcal{A}}(2^{k+1}m-3) \\ f_{\mathcal{A}}(2^{k+1}m-4) \end{pmatrix} = \begin{pmatrix} f_{\mathcal{A}}(2^k m) + f_{\mathcal{A}}(2^k m-2) \\ f_{\mathcal{A}}(2^k m-1) + f_{\mathcal{A}}(2^k m-2) \\ f_{\mathcal{A}}(2^k m-1) + f_{\mathcal{A}}(2^k m-3) \\ f_{\mathcal{A}}(2^k m-2) + f_{\mathcal{A}}(2^k m-3) \\ f_{\mathcal{A}}(2^k m-2) + f_{\mathcal{A}}(2^k m-4) \end{pmatrix}.$$

and $M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ satisfies $\omega_{k+1}(m) = M\omega_k(m)$.

## Theorem

*Let $\mathcal{A}, f_{\mathcal{A}}(n), M,$ and $\omega_k(m)$ be as above, with the additional assumption that there exists some odd $a_i \in \mathcal{A}$. Define*

$$s_{\mathcal{A}}(r) = \sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n).$$

*Let $|\mathcal{A}|$ denote the number of elements in the set $\mathcal{A}$. Then*

$$\lim_{r\to\infty} \frac{s_{\mathcal{A}}(r)}{|\mathcal{A}|^r} = c(\mathcal{A}),$$

*where $c(\mathcal{A}) \in \mathbb{Q}$, so*

$$s_{\mathcal{A}}(r) \approx c(\mathcal{A})\,|\mathcal{A}|^r.$$

# Example: $\mathcal{A} = \{0, 2, 3\}$

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 1)$$

$$f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell - 1)$$

$$\begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m - 1) \\ f_{\mathcal{A}}(2^{k+1}m - 2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_{\mathcal{A}}(2^{k}m) \\ f_{\mathcal{A}}(2^{k}m - 1) \\ f_{\mathcal{A}}(2^{k}m - 2) \end{pmatrix}$$

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The characteristic polynomial of $M$ is $g(x) = -(x - 1)(x^2 - x - 1)$.

## $\{0, 2, 3\}$ continued

$$
\begin{aligned}
s_{\mathcal{A}}(r) &= \sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n) \\
&= \sum_{n=2^{r-1}}^{2^r-1} \left(f_{\mathcal{A}}(2n) + f_{\mathcal{A}}(2n+1)\right) \\
&= \sum_{n=2^{r-1}}^{2^r-1} \left(f_{\mathcal{A}}(n) + f_{\mathcal{A}}(n-1) + f_{\mathcal{A}}(n-1)\right) \\
&= s_{\mathcal{A}}(r-1) + 2\sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n-1)
\end{aligned}
$$

$$
\begin{aligned}
s_{\mathcal{A}}(r) &= s_{\mathcal{A}}(r-1) + 2\sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n) + 2f_{\mathcal{A}}(2^{r-1}-1) - 2f_{\mathcal{A}}(2^r-1) \\
&= 3s_{\mathcal{A}}(r-1) + 2f_{\mathcal{A}}(2^{r-1}-1) - 2f_{\mathcal{A}}(2^r-1) \\
&= 3s_{\mathcal{A}}(r-1) + 2F_{r-2} - 2F_{r-1} \\
&= 3s_{\mathcal{A}}(r-1) - 2F_{r-3}
\end{aligned}
$$

- Solution to homogeneous recurrence relation

$$s_{\mathcal{A}}(r) = c_1 3^r$$

- Solution to inhomogeneous recurrence relation

$$s_{\mathcal{A}}(r) = c_1 3^r + c_2 \phi^r + c_3 \bar{\phi}^r + c_4 (1)^r$$

$$\begin{aligned} s_{\mathcal{A}}(r+2) - s_{\mathcal{A}}(r+1) - s_{\mathcal{A}}(r) &= c_1 3^r(3^2 - 3 - 1) + c_2\phi^r(\phi^2 - \phi - 1) \\ &\quad + c_3\bar{\phi}^r(\bar{\phi}^2 - \bar{\phi} - 1) + c_4(1^2 - 1 - 1) \\ &= c_1 3^r \cdot 5 - c_4 \end{aligned}$$

$$
\begin{aligned}
s_{\mathcal{A}}(r+2) - s_{\mathcal{A}}(r+1) - s_{\mathcal{A}}(r) &= c_1 3^r(3^2 - 3 - 1) + c_2\phi^r(\phi^2 - \phi - 1) \\
&\quad + c_3\bar{\phi}^r(\bar{\phi}^2 - \bar{\phi} - 1) + c_4(1^2 - 1 - 1) \\
&= c_1 3^r \cdot 5 - c_4
\end{aligned}
$$

We can plug in $r = 0$ and $r = 1$ and compute sums to solve and find that $c_1 = \frac{2}{5}$. Hence

$$
\lim_{r\to\infty} \frac{s_{\mathcal{A}}(r)}{|\mathcal{A}|^r} = \lim_{r\to\infty} \frac{s_{\{0,2,3\}}(r)}{4^r} = \frac{2}{5}.
$$

# Proof

Let $g(\lambda) := \det(M - \lambda I)$ be the characteristic polynomial of $M$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_y$, where each $\lambda_i$ has multiplicity $e_i$, so

$$g(\lambda) = \sum_{k=0}^{a_z+1} \alpha_k \lambda^k.$$

By Cayley-Hamilton, we know that $g(M) = 0$. Thus we have

$$0 = g(M) = \sum_{k=0}^{a_z+1} \alpha_k M^k$$

and hence, for all $r$,

$$0 = \left( \sum_{k=0}^{a_z+1} \alpha_k M^k \right) \omega_r(m) = \sum_{k=0}^{a_z+1} \alpha_k \omega_{r+k}(m).$$

Let $I_r = \{2^r, 2^r + 1, 2^r + 2, \ldots, 2^{r+1} - 1\}$. Then $I_r = 2I_{r-1} \cup (2I_{r-1} + 1)$. Thus

$$\begin{aligned} s_{\mathcal{A}}(r) &= \sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n) \\ &= \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(2n) + f_{\mathcal{A}}(2n+1) \\ &= \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n) + f_{\mathcal{A}}(n-b_2) + \cdots + f_{\mathcal{A}}(n-b_s) \\ &\qquad + f_{\mathcal{A}}(n-c_1) + \cdots + f_{\mathcal{A}}(n-c_t). \end{aligned}$$

Now

$$\sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n-k) = \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n) + \sum_{j=1}^{k} \left(f_{\mathcal{A}}(2^{r-1}-j) - f(2^r-j)\right),$$

so

$$s_{\mathcal{A}}(r) = |\mathcal{A}| \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n) + h(r) = |\mathcal{A}|\, s_{\mathcal{A}}(r-1) + h(r),$$

where $h_r$ is such that

$$\sum_{k=0}^{a_z+1} \alpha_k h(r+k) = 0.$$

The solution to this inhomogeneous recurrence relation is of the form

$$s_{\mathcal{A}}(r) = c_1 \left|\mathcal{A}\right|^r + \sum_{i=1}^{y} p_i(\lambda_i),$$

where $p_i(\lambda_i) = \sum_{j=1}^{e_i} c_{ij} r^{j-1} \lambda_i^r$.

We can compute $\sum_{k=0}^{a_z+1} \alpha_k s_{\mathcal{A}}(r+k)$, and for sufficiently large $r$, we have

$$\sum_{k=0}^{a_z+1} \alpha_k s_{\mathcal{A}}(r+k) = c_1 \sum_{k=0}^{a_z+1} \alpha_k \left|\mathcal{A}\right|^{r+k} + 0 = c_1 \left|\mathcal{A}\right|^r g\left(\left|\mathcal{A}\right|\right).$$

Then we can solve for $c_1$ to see that

$$c_1 = \frac{\sum_{k=0}^{a_z+1} \alpha_k s_{\mathcal{A}}(r+k)}{\left|\mathcal{A}\right|^r g\left(\left|\mathcal{A}\right|\right)}.$$

| $\mathcal{A}$ | $c(\mathcal{A})$ | $N(c(\mathcal{A}))$ | $\mathcal{A}$ | $c(\mathcal{A})$ | $N(c(\mathcal{A}))$ |
|---|---|---|---|---|---|
| $\{0,1,2\}$ | $1$ | 1.000 | $\{0,1,3\}$ | $\frac{4}{5}$ | 0.800 |
| $\{0,1,4\}$ | $\frac{5}{8}$ | 0.625 | $\{0,1,5\}$ | $\frac{14}{25}$ | 0.560 |
| $\{0,1,6\}$ | $\frac{425}{852}$ | 0.499 | $\{0,1,7\}$ | $\frac{176}{391}$ | 0.450 |
| $\{0,1,8\}$ | $\frac{137}{338}$ | 0.405 | $\{0,1,9\}$ | $\frac{1448}{3775}$ | 0.384 |
| $\{0,1,10\}$ | $\frac{1990}{5527}$ | 0.360 | $\{0,1,11\}$ | $\frac{3223}{9476}$ | 0.340 |
| $\{0,1,12\}$ | $\frac{2020}{6283}$ | 0.322 | $\{0,1,13\}$ | $\frac{47228}{154123}$ | 0.306 |
| $\{0,1,14\}$ | $\frac{35624}{122411}$ | 0.291 | $\{0,1,15\}$ | $\frac{699224}{2501653}$ | 0.280 |

| $\mathcal{A}$ | $c(\mathcal{A})$ | $\tilde{\mathcal{A}}$ | $c(\tilde{\mathcal{A}})$ |
|---|---|---|---|
| $\{0,1,2,4\}$ | $\frac{7}{11}$ | $\{0,2,3,4\}$ | $\frac{3}{11}$ |
| $\{0,2,3,6\}$ | $\frac{2531}{9536}$ | $\{0,3,4,6\}$ | $\frac{1344}{9536}$ |
| $\{0,1,6,9\}$ | $\frac{3401207}{16513920}$ | $\{0,3,8,9\}$ | $\frac{1156032}{16513920}$ |
| $\{0,1,7,9\}$ | $\frac{132416}{655040}$ | $\{0,2,8,9\}$ | $\frac{51145}{655040}$ |
| $\{0,4,5,6,9\}$ | $\frac{4044}{83753}$ | $\{0,3,4,5,9\}$ | $\frac{6716}{83753}$ |

## Theorem

*Let* $\mathcal{A}, f_{\mathcal{A}}(n)$ *and* $M = [m_{\alpha,\beta}]$ *be as above. Define*

$$\tilde{\mathcal{A}} := \{0, a_z - a_{z-1}, \ldots, a_z - a_1, a_z\}.$$

### Theorem

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*Let $N = [n_{\alpha,\beta}]$ be the $(a_z + 1) \times (a_z + 1)$ matrix such that*

$$\begin{pmatrix} f_{\tilde{\mathcal{A}}}(2n) \\ f_{\tilde{\mathcal{A}}}(2n-1) \\ \vdots \\ f_{\tilde{\mathcal{A}}}(2n-a_z) \end{pmatrix} = N \begin{pmatrix} f_{\tilde{\mathcal{A}}}(n) \\ f_{\tilde{\mathcal{A}}}(n-1) \\ \vdots \\ f_{\tilde{\mathcal{A}}}(n-a_z) \end{pmatrix}.$$

## Theorem

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*Then $m_{\alpha,\beta} = n_{a_z-\alpha, a_z-\beta}$.*

# Proof

Recall we can write

$$\mathcal{A} := \{0, 2b_1, \ldots, 2b_s, 2c_1 + 1, \ldots, 2c_t + 1\},$$

so that

$$f_{\mathcal{A}}(2n - 2j) = f_{\mathcal{A}}(n - j) + f_{\mathcal{A}}(n - j - b_1) + \cdots + f_{\mathcal{A}}(n - j - b_s)$$

and

$$f_{\mathcal{A}}(2n - 2j - 1) = f_{\mathcal{A}}(n - j - c_1 - 1) + \cdots + f_{\mathcal{A}}(n - j - c_t - 1)$$

for $j$ sufficiently large.

Then $m_{\alpha,\beta} = 1$

$$\iff f_{\mathcal{A}}(n-\beta) \text{ is a summand in the recursive sum that expresses } f_{\mathcal{A}}(2n-\alpha)$$

$$\iff 2n - \alpha = 2(n-\beta) + K, \text{ where } K \in \mathcal{A}$$

$$\iff 2\beta - \alpha \in \mathcal{A}.$$

Now $n_{a_z-\alpha,a_z-\beta} = 1$

$$\begin{aligned} &\iff f_{\tilde{\mathcal{A}}}(n-(a_z-\beta)) \text{ is a summand in the recursive sum} \\ &\qquad \text{that expresses } f_{\tilde{\mathcal{A}}}(2n-(a_z-\alpha)) \\ &\iff 2n-(a_z-\alpha) = 2(n-(a_z-\beta)) + \tilde{K}, \text{ where } \tilde{K} \in \tilde{\mathcal{A}} \\ &\iff a_z+\alpha-2\beta = \tilde{K} \\ &\iff 2\beta-\alpha \in \mathcal{A}. \end{aligned}$$

Thus $M = A^{-1}NA$, where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

so $M$ and $N$ have the same characteristic polynomial. Hence the denominator of $c(\mathcal{A})$ is the same as the denominator of $c(\tilde{\mathcal{A}})$.

# Future Research Ideas

- Finding more families of robust polynomials
- Determining the cluster points of

$$\left\{ \frac{\beta_1(f)}{\beta_0(f) + \beta_1(f)} : f(x) \text{ is a polynomial} \right\}$$

- Finding formulas for $c(\mathcal{A})$
- Exploring properties of $f_{\mathcal{A}}(n)$ in bases other than 2

Let $f(x)$ be an element of $\mathbb{F}_2[x]$ with $\deg(f(x)) = k$. Then Lidl & Niederreiter's *Finite Fields* gives an upper bound of

$$|\beta_1(f(x)) - \beta_0(f(x))| \leq 2^{k/2}.$$

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Thus

$$|\beta_1(f_{3,1}(x)) - \beta_0(f_{3,1}(x))| = 37 - 26 = 11 \leq 2^{9/2} \approx 22.6$$

and

$$|\beta_1(f_{3,2}(x)) - \beta_0(f_{3,2}(x))| = 45 - 28 = 17 \leq 2^{10/2} = 32.$$

In general,

$$\begin{aligned}|\beta_1(f_{r,1}(x)) - \beta_0(f_{r,1}(x))| &= 4^r - 3^r - (3^r - 1) \\ &= 4^r - 2 \cdot 3^r + 1 \\ &\ll 2^{\frac{1}{2}(2^r+1)} \\ &= 4^{2^{r-2}+\frac{1}{4}}\end{aligned}$$

and

$$\begin{aligned}|\beta_1(f_{r,2}(x)) - \beta_0(f_{r,2}(x))| &= 4^r - 3^r + 2^r - (3^r + 1) \\ &= 4^r - 2 \cdot 3^r + 2^r - 1 \\ &\ll 2^{\frac{1}{2}(2^r+2)} \\ &= 4^{2^{r-2}+\frac{1}{2}}.\end{aligned}$$

# Example: $\mathcal{A} = \{0, 1, 4\}$

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 2)$$

$$f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell)$$

$$\begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m-1) \\ f_{\mathcal{A}}(2^{k+1}m-2) \\ f_{\mathcal{A}}(2^{k+1}m-3) \\ f_{\mathcal{A}}(2^{k+1}m-4) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{\mathcal{A}}(2^{k}m) \\ f_{\mathcal{A}}(2^{k}m-1) \\ f_{\mathcal{A}}(2^{k}m-2) \\ f_{\mathcal{A}}(2^{k}m-3) \\ f_{\mathcal{A}}(2^{k}m-4) \end{pmatrix}$$

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The characteristic polynomial of $M$ is $g(x) = -(x - 1)^4(x + 1)$.

## $\{0, 1, 4\}$ continued

$$\begin{aligned} s_{\mathcal{A}}(r) &= \sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n) \\ &= \sum_{n=2^{r-1}}^{2^r-1} \left(f_{\mathcal{A}}(2n) + f_{\mathcal{A}}(2n+1)\right) \\ &= \sum_{n=2^{r-1}}^{2^r-1} \left(f_{\mathcal{A}}(n) + f_{\mathcal{A}}(n-2) + f_{\mathcal{A}}(n)\right) \\ &= 2s_{\mathcal{A}}(r-1) + \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n-2) \\ &= 3s_{\mathcal{A}}(r-1) + f_{\mathcal{A}}(2^{r-1}-2) + f_{\mathcal{A}}(2^{r-1}-1) \\ &\quad - f_{\mathcal{A}}(2^r-2) - f_{\mathcal{A}}(2^r-1) \end{aligned}$$

- Solution to homogeneous recurrence relation

$$s_{\mathcal{A}}(r) = c_1 3^r$$

- Solution to inhomogeneous recurrence relation

$$s_{\mathcal{A}}(r) = c_1 3^r + c_2(-1)^r + c_3(1)^r + c_4 r(1)^r + c_5 r^2(1)^r + c_6 r^3(1)^r$$

- Hence

$$\lim_{r\to\infty} \frac{s_{\mathcal{A}}(r)}{|\mathcal{A}|^r} = c_1.$$