Balinski's theorem and Regularity of Line Arrangements

Bruno Benedetti
(University of Miami)

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Joint work with

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- Michela di Marca, Matteo Varbaro (U Genova), 2016

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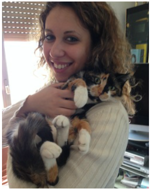

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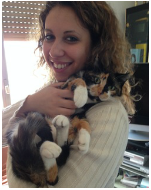

Warming up: Linear Optimization in five minutes

Given a **linear** function $f : \mathbb{R}^d \to \mathbb{R}$, and a region $P \subset \mathbb{R}^d$, suppose we want to find $\max\{f(\underline{x}) : \underline{x} \in P\}$.

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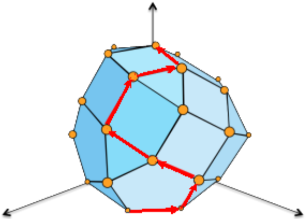

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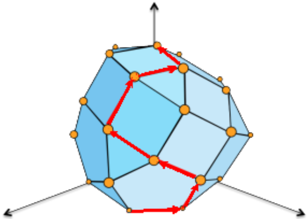

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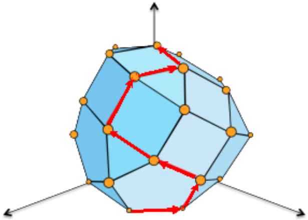

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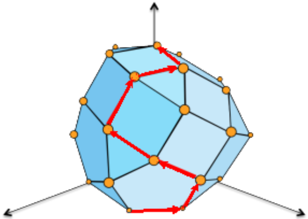

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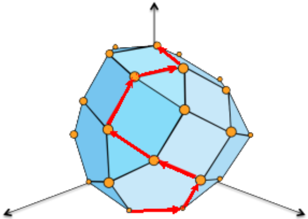

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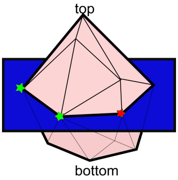

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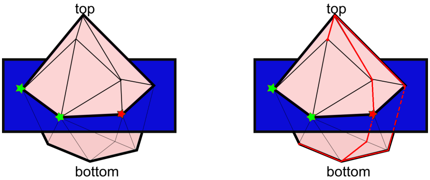

Plan for today

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Part I. Many Classes of Dual Graphs.

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Part III. (time permitting) Arrangements of Curves.

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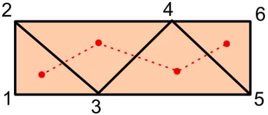

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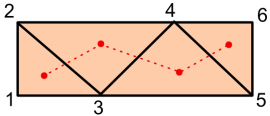

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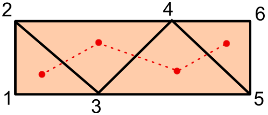

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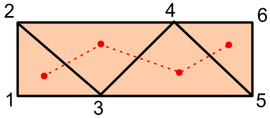

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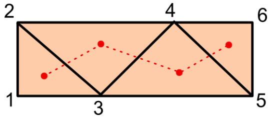

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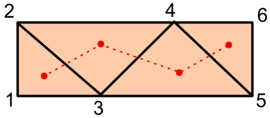

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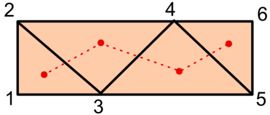

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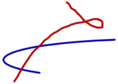

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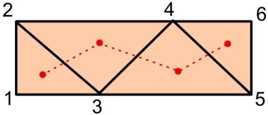

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  Vertices correspond to the irreducible components $C_1, \ldots, C_s$. (Equidimensional means, they all have same dimension.) We put an edge between two distinct vertices, if and only if the corresponding components intersect in dimension one less.

Dual graphs of curves = dual graphs of varieties

By intersecting a d-dimensional object in $\mathbb{P}^n$ with a generic hyperplane, we get an object in $\mathbb{P}^{n-1}$ with dimension $d-1$, and same dual graph!

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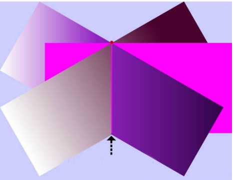

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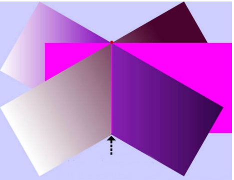

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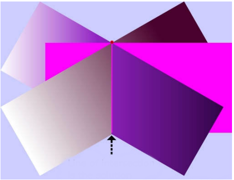

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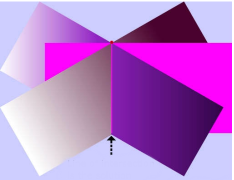

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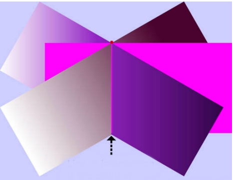

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Attention!, graphs like

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So two options!, not three. So some of the three lines r_5, r_6, r_7 have to coincide.

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Try drawing it. Let $P = r_1 \cap r_2$ and let $Q = r_3 \cap r_4$. Let p be the plane containing $r_1 \cup r_2$, and let q be the plane containing $r_3 \cup r_4$.

How can a line meet all four r_1, r_2, r_3, r_4? There are only two chances (possibly coinciding):

- either it's the line through P and Q, or
- it's the line of intersection of the planes $p \cap q$

So two options!, not three. So some of the three lines r_5, r_6, r_7 have to coincide. a contradiction

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Kollar 2012: every graph is dual to some arrangement of **curves**.

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It remains to see how dual graphs of **simpl. complexes** fit the hierarchy.

Simplicial Complexes, Seen as Varieties (Stanley-Reisner)

Definition by example:

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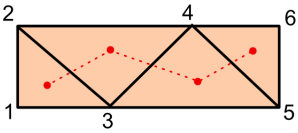

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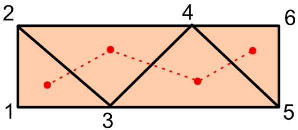

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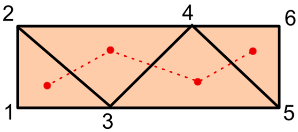

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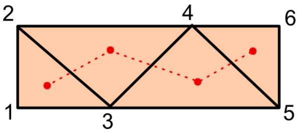

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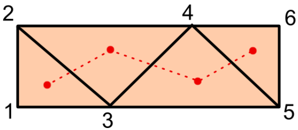

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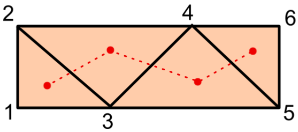

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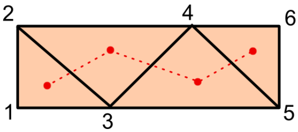

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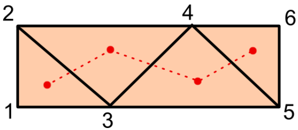

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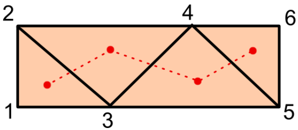

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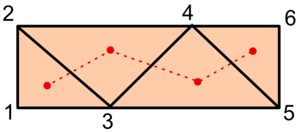

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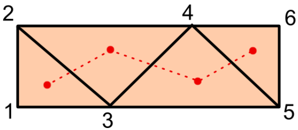

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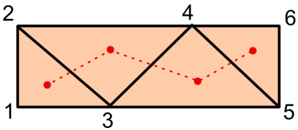

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... So dual graph of $V(I_\Delta)$ is same of Δ.

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$$\text{This implies} \left\{ \begin{array}{c} \text{dual graphs of} \\ \text{simplicial complexes} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{dual graphs} \\ \text{of lines} \end{array} \right\}.$$

(Graphs like $\{12, 13, 15, 23, 24, 34, 45\}$ show the containment is strict.)

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Part II. The Algebraic Machinery (sketch).

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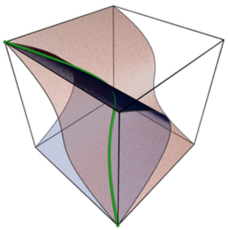
Complete intersections are the varieties for which "=" holds.



The "twisted cubic" (s^3, s^2t, st^2, t^3) of $\mathbb{P}^3$ is not a complete intersection: one needs at least three (hyper)surfaces to cut it out.

Liaison theory and Gorenstein-ness

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- Note: liaison theory (and the isomorphism above!) works also under a weaker assumption than "complete intersection", called "Gorenstein".
- Among Stanley-Reisner varieties, this Gorenstein property has been nicely explained by Stanley: "S/I_Δ Gorenstein iff Δ is the join of a homology sphere with a simplex".

Regularity for Algebraists

Given a projective scheme X in $\mathbb{P}^n$, $\exists!$ saturated homogeneous ideal $I_X \subset S := \mathbb{K}[x_0, \ldots, x_n]$ s. t. $X = \text{Proj}(S/I_X)$; one says X is **aCM** (resp. **aG**) if S/I_X is Cohen–Macaulay (resp. Gorenstein).

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Balinski, Klee (1975)

The dual graph of every $(d-1)$-dimensional triangulated homology sphere (or manifold) is d-regular and d-connected.

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Special case 2: if X is a complete intersection. ($\operatorname{reg} X$ is the sum of the degree of the components, minus their number, minus 1.)

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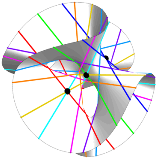

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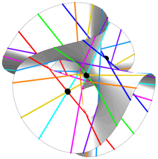

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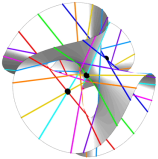

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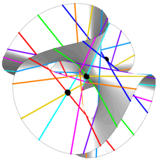

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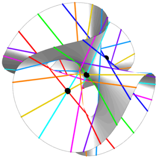

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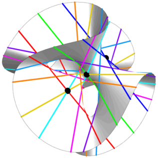

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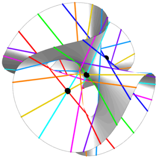

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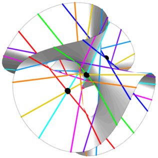

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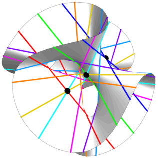

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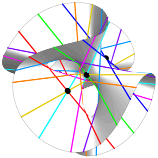

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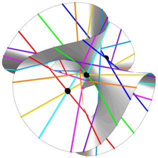

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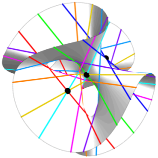

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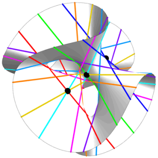

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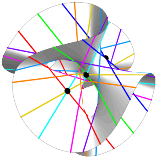

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We proved X is a complete intersection, with $a = 3$ and $b = 4$: as our Corollary claims, every line intersects **exactly** 5 other lines.

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Part III. From Lines to Curves.

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Order of the prime ideals as you wish. Let $I_B = \mathfrak{p}_1 \cap \ldots \mathfrak{p}_{r-1}$ and $I_A = \mathfrak{p}_r \cap \ldots \cap \mathfrak{p}_s$. Want to prove that $G(I_A)$ is connected.

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4. So $H^1_{\mathfrak{m}}(S/I_A)_0 = 0$. This implies that $H^0(C_A, \mathcal{O}_{C_A}) \cong \mathbb{K}$, which in turn implies that C_A is a connected curve.