Forbidden minors for projective planarity

Guoli Ding
Louisiana State University

Joint work with Perry Iverson and Kimberly D'Souza

CombinaTexas, May 7, 2016

Background

- 1930 Kuratowski:

  planar $\Leftrightarrow$ no $\{K_5, K_{3,3}\}$-subdivision

- 1930^+ Erdös: what about other surfaces?

  For any surface Σ,

  let $\mathcal{S}_\Sigma = \{$minimal non-embeddable graphs$\}$.

  Note: Σ-embeddable $\Leftrightarrow$ no $\mathcal{S}_\Sigma$-subdivision

  Is $\mathcal{S}_\Sigma$ finite?

Background

- 1978 Glover-Huneke: $\mathcal{S}_{\mathbb{N}_1}$ is finite
- 1980 Archdeacon: $|\mathcal{S}_{\mathbb{N}_1}| = 103$
- 1989 Archdeacon-Huneke: $\mathcal{S}_{\mathbb{N}_k}$ is finite $(\forall k)$
- 1990 Robertson-Seymour: $\mathcal{S}_\Sigma$ is finite $(\forall \Sigma)$
- 1990 Everyone: what are the graphs in each $\mathcal{S}_\Sigma$?

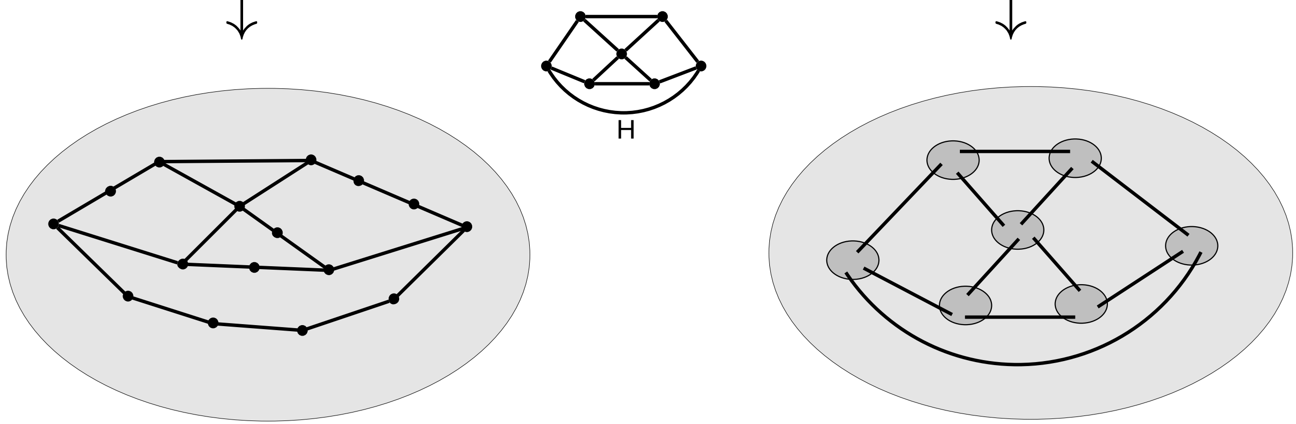
is this the right question to ask ?

Remarks on Robertson-Seymour

(1) G contains H

as a subdivision vs as a minor

↓ ↓



(2) $\forall H \; \exists H_1, H_2, \ldots, H_k$ such that

$$H \leq_m G \iff H_i \leq_s G \text{ for some } i.$$

Remarks on Robertson-Seymour

(3) Robertson-Seymour:

$$\mathcal{M}_\Sigma = \{\text{minor-minimal non-embeddable graphs}\}$$

is finite, for every Σ.

(4) Consequently, $\mathcal{S}_\Sigma$ is finite, for every Σ.

Since $|\mathcal{M}_\Sigma| \leq |\mathcal{S}_\Sigma|$, we will talk about $\mathcal{M}_\Sigma$, instead of $\mathcal{S}_\Sigma$.

Problem. What are the graphs in each $\mathcal{M}_\Sigma$?

Known results:

- $\mathcal{M}_{\mathbb{S}_0} = \{K_5, K_{3,3}\}$
- $|\mathcal{M}_{\mathbb{N}_1}| = 35$
- $|\mathcal{M}_{\mathbb{S}_1}| \geq 16,629$

Not all graphs in $\mathcal{M}_\Sigma$ are equally important!

- some are of low connectivity – a major defect!
- some are "accident"

Theorem (Archdeacon)

A graph is projective planar iff it does not contain any of the following 35 as a minor:

(0) any 0-sum of two graphs in $\{K_5, K_{3,3}\}$

(1) any 1-sum of two graphs in $\{K_5, K_{3,3}\}$

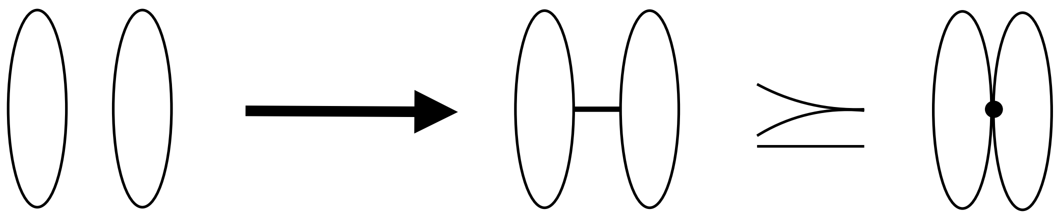
(2) any 2-sum of two graphs in $\{K_5, K_{3,3}\}$

(3) another 23 3-connected graphs

Let $\mathcal{A} = \mathcal{M}_{\mathbb{N}_1}$ be the set of 35 Archdeacon graphs.

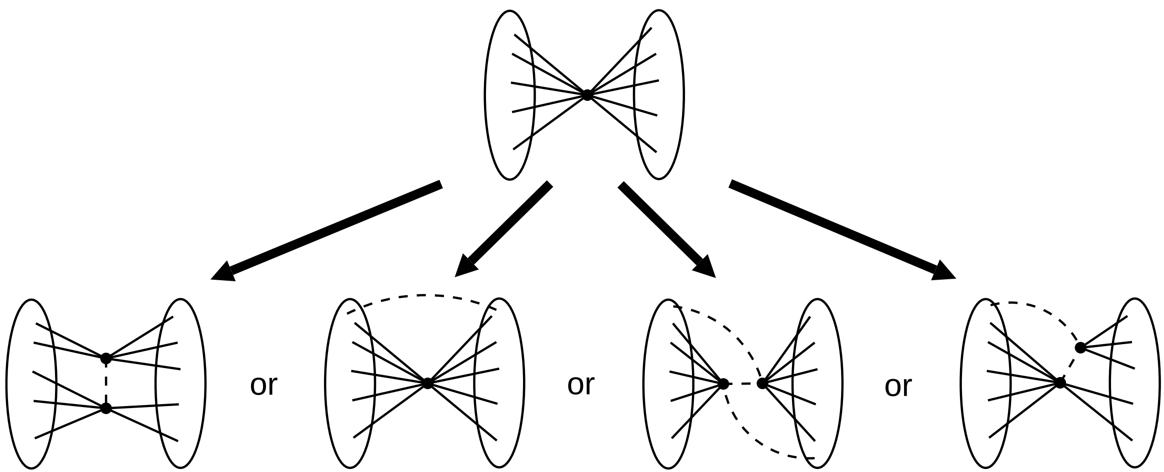
Proposition 1. Let $\mathcal{A}_1$ be the 32 connected graphs in $\mathcal{A}$. Then a connected graph G is projective iff G does not contain any graph in $\mathcal{A}_1$ as a minor.

Proof. Let G be connected with $G \succeq H$.



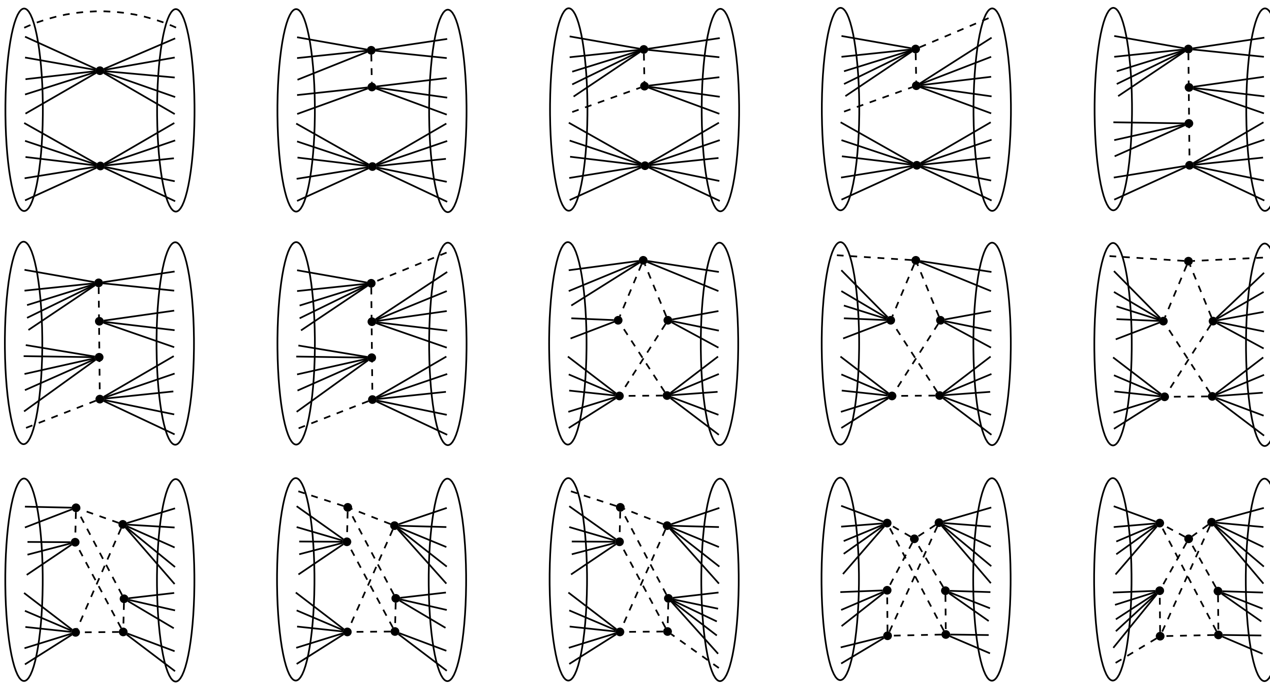
Proposition 2. Let $\mathcal{A}_2$ be the 29 2-connected graphs in $\mathcal{A}$. Then a 2-connected graph G is projective iff G does not contain any graph in $\mathcal{A}_2$ as a minor.

Proof. Let G be 2-connected with $G \succeq H$.



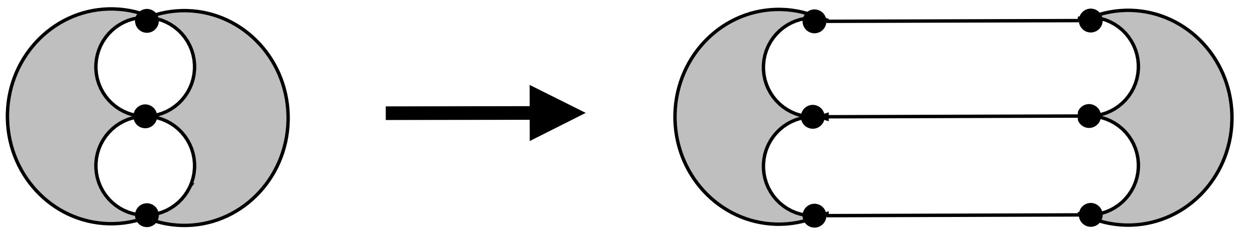
Proposition 3. Let $\mathcal{A}_3$ be the 23 3-connected graphs in $\mathcal{A}$. Then a 3-connected graph G is projective iff G does not contain any graph in $\mathcal{A}_3$ as a minor.

Proof. Let G be 3-connected with $G \succeq H$.



Suppose:

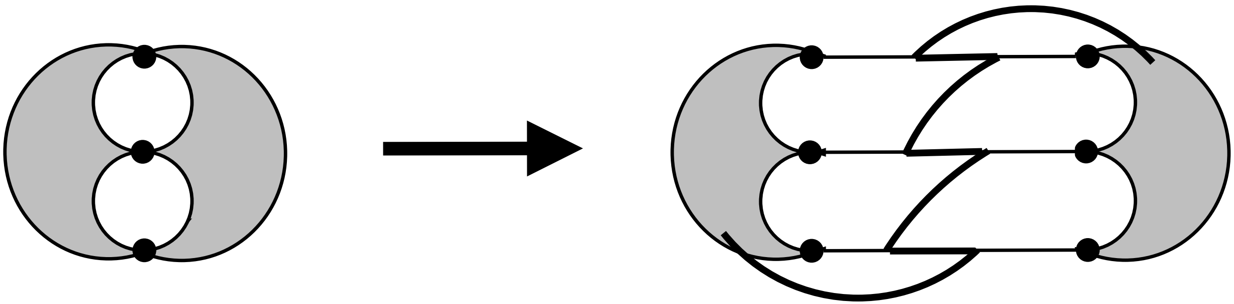
- H is a minor of G, and
- a k-separation of H does not extend to G



H $\quad$ H in G

Suppose:

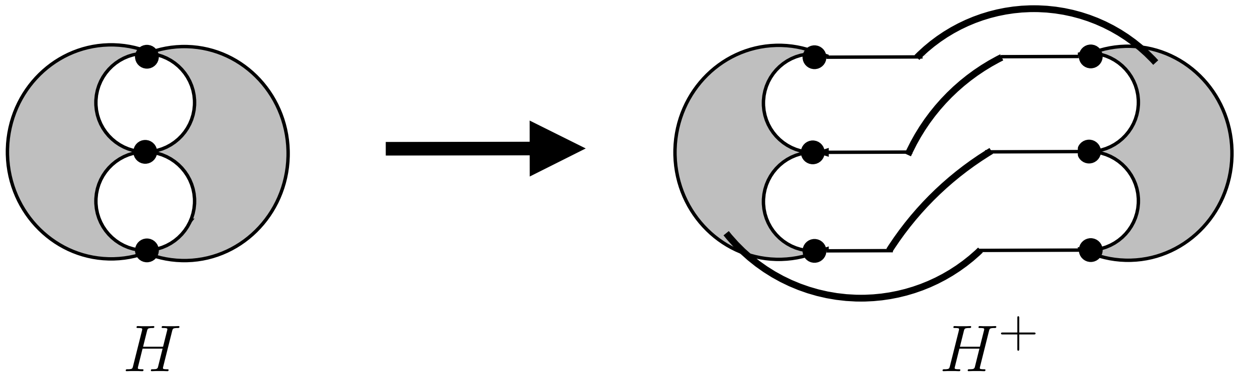
- H is a minor of G, and
- a k-separation of H does not extend to G



H $\qquad$ $H+$ augmenting path

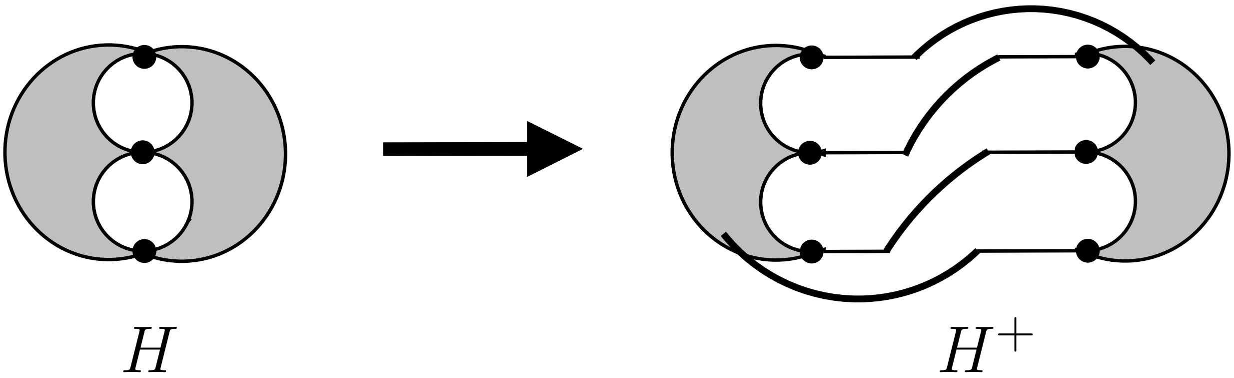
Suppose:

- H is a minor of G, and
- a k-separation of H does not extend to G



Suppose:

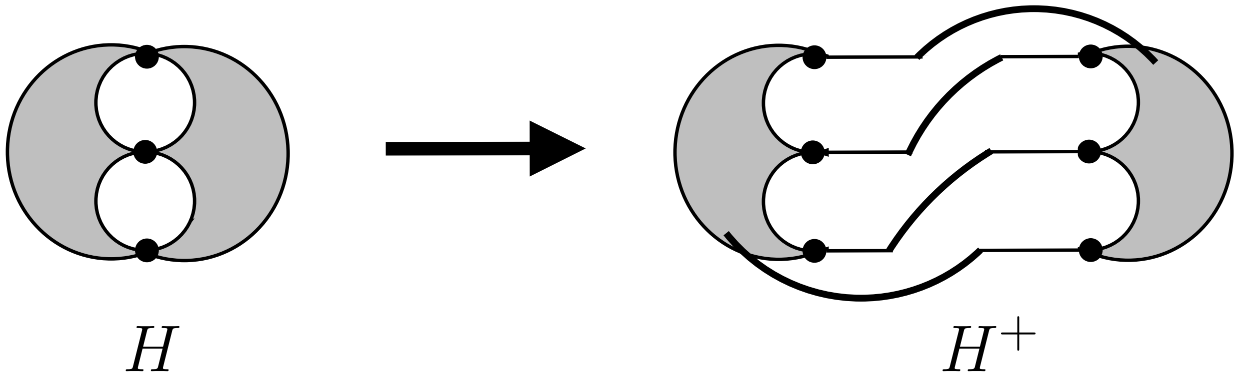
- H is a minor of G, and
- a k-separation of H does not extend to G



Lemma. G contains H^+.

Suppose:

- H is a minor of G, and
- a k-separation of H does not extend to G



Lemma. G contains H^+.

This Lemma gives us a short proof for Proposition 3:

3-connected $\mathcal{A}_3$-free graphs are projective

Proof. We need only prove that every 3-connected non-projective graph contains a graph in $\mathcal{A}_3$ as a minor. By Theorem 2, we may assume that G has a graph $A \in \mathcal{A}_2$ as a minor, where A is one of the six graphs in $\mathcal{A}_2$ of connectivity two, which are listed in Figure 2.1. Notice that each of these graphs is a 2-sum of two graphs among $\{K_{3,3}, K_5\}$. By Theorem 2, G contains a twist J of the 2-separation of A as a minor where J is constructed from rooted graphs (J_i, R_i) $(i = 1, 2)$ that are among the graphs shown in Figure 1, which we call $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, $K_{3,3}^{N3}$, $K_{3,3}^{E1}$, $K_{3,3}^{E2}$, K_5^1, and K_5^2, respectively. Let M be the matching used to construct J from J_1 and J_2.

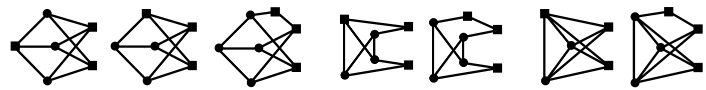


Figure 1: Seven possibilities for (J_i, R_i): $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, $K_{3,3}^{N3}$, $K_{3,3}^{E1}$, $K_{3,3}^{E2}$, K_5^1, and K_5^2

First assume (J_1, R_1) is one of $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, or $K_{3,3}^{N3}$, and contract the entire matching M to obtain J'. Notice that $K_{3,3}^{N3}$ can be contracted to $K_{3,3}^{N2}$, $K_{3,3}^{E2}$ can be contracted to $K_{3,3}^{E1}$, and K_5^2 can be contracted to K_5^1. So we may assume (J_1, R_1) is either $K_{3,3}^{N1}$ or $K_{3,3}^{N2}$ and (J_2, R_2) is one of $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, $K_{3,3}^{E1}$, or K_5^1. Now notice that $K_{2,3}$ rooted at the three mutually non-adjacent vertices can be obtained by contracting and deleting edges of $K_{3,3}^{N2}$, $K_{3,3}^{E1}$, or K_5^1. Therefore if (J_1, R_1) or (J_2, R_2) is $K_{3,3}^{N1}$, then J' contains $K_{3,5} = E_3 \in \mathcal{A}_3$ as a minor. Now we may assume that (J_1, R_1) is $K_{3,3}^{N2}$ and (J_2, R_2) is $K_{3,3}^{N2}$, $K_{3,3}^{E1}$, or K_5^1. If (J_2, R_2) is $K_{3,3}^{N2}$, delete an edge from it to obtain $K_{3,3}^{E1}$; if (J_2, R_2) is $K_{3,3}^{E1}$, J' has either $E_5 \in \mathcal{A}_3$ or $F_1 \in \mathcal{A}_3$ as a subgraph; and if (J_2, R_2) is K_5^1, J' has $D_3 \in \mathcal{A}_3$ as a subgraph.

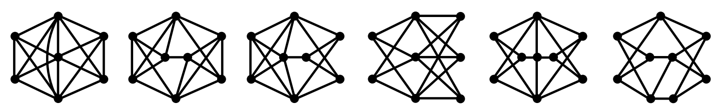


Figure 2: Six graphs in $\mathcal{A}_3$: B_1, C_7, D_3, E_3, E_5, and F_1

Now (J_i, R_i) must be among $K_{3,3}^{E1}$, $K_{3,3}^{E2}$, K_5^1, and K_5^2 for $i = 1, 2$. Suppose (J_1, R_1) is $K_{3,3}^{E2}$ or K_5^2. We contract the entire matching M to obtain J'. If (J_2, R_2) is $K_{3,3}^{E2}$ or K_5^2, contract it to $K_{3,3}^{E1}$ or K_5^1, respectively. In case (J_1, R_1) is $K_{3,3}^{E2}$, if (J_2, R_2) is $K_{3,3}^{E1}$, J' has F_1 as a minor, and if (J_2, R_2) is K_5^1, J' has D_3 as a minor. In case (J_1, R_1) is K_5^2, if (J_2, R_2) is $K_{3,3}^{E1}$, J' has D_3 or F_1 as a minor, if (J_2, R_2) is K_5^1, J' has $C_7 \in \mathcal{A}_3$ as a subgraph.

So (J_i, R_i) is either $K_{3,3}^{E1}$ or K_5^1 for $i = 1, 2$. In this case, we may no longer contract the entire matching M since this may result in a projective graph. Suppose $\{v_1, v_2\}$ is the 2-cut of A, then contract any edge of M incident to some vertex with label either v_1 or v_2. Then if (J_1, R_1) and (J_2, R_2) are both $K_{3,3}^{E1}$, J' has either E_5 or F_1 as a subgraph. If (J_1, R_1) is $K_{3,3}^{E1}$ and (J_2, R_2) is K_5^1, J' has D_3 as a subgraph. Finally if (J_1, R_1) and (J_2, R_2) are both K_5^1, J' has either B_1 or C_7 as a subgraph. QED

Theorem.

(1) A connected graph is projective iff it is $\mathcal{A}_1$-free.

(2) A 2-connected graph is projective iff it is $\mathcal{A}_2$-free.

(3) A 3-connected graph is projective iff it is $\mathcal{A}_3$-free.

(4) An internally 4-connected graph is projective iff it is $\mathcal{A}_4^*$-free.

our first main result

proved by Robertson, Seymour, and Thomas

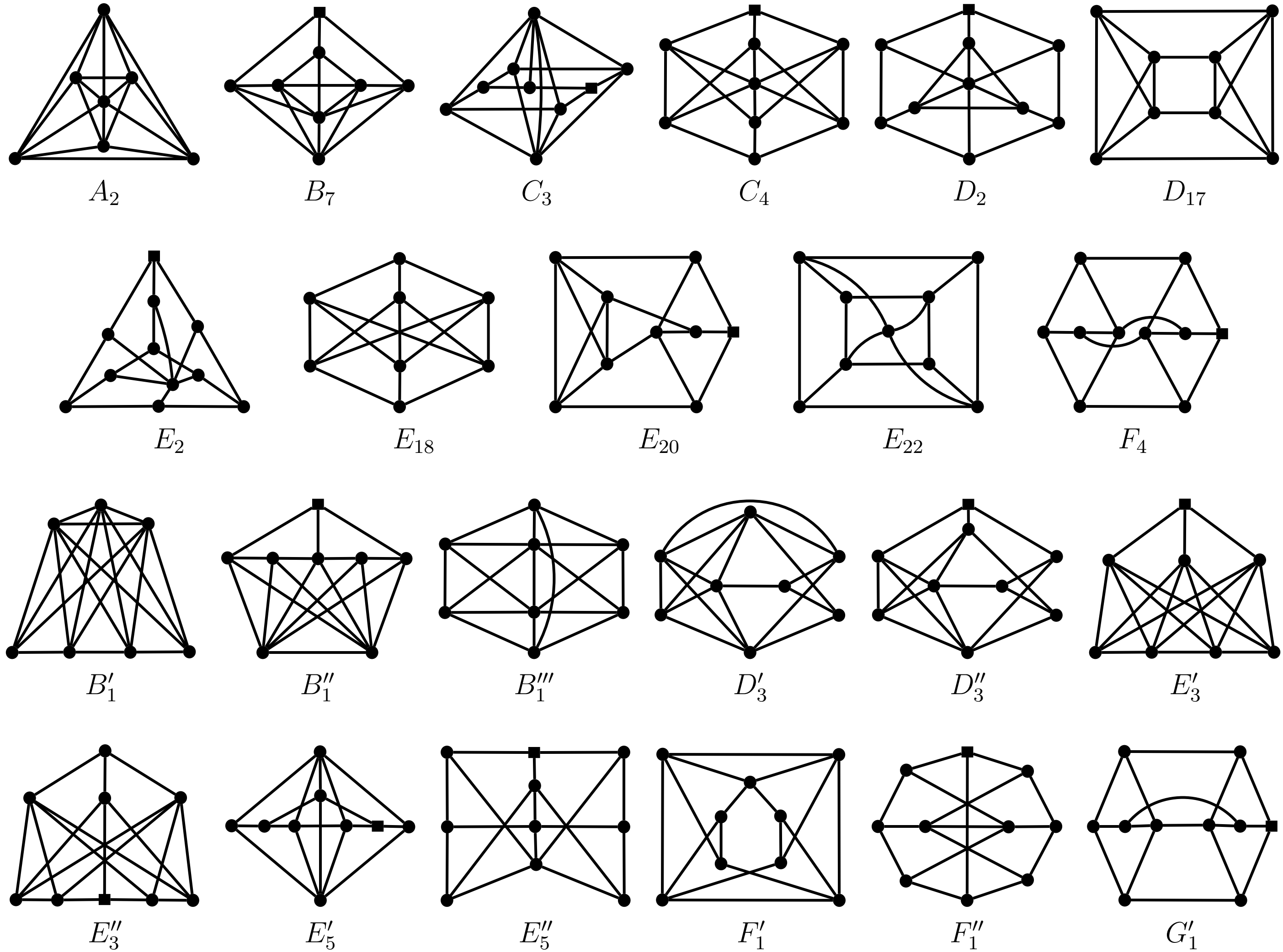
Proof of (4).

$$\mathcal{A}_3 = \mathcal{A}_4 \ \cup \ \underbrace{\{B_1, C_7, D_3, D_9, D_{12} \\ E_3, E_5, E_{11}, E_{19}, E_{27}, F_1, G_1\}}_{\text{12 graphs}}$$

$$\Downarrow \quad \text{(Lemma)}$$

$$\mathcal{A}_4^*$$

which are . . . . . .


A_2
B_7
C_3
C_4
D_2
D_{17}
E_2
E_{18}
E_{20}
E_{22}
F_4
B_1'
B_1''
B_1'''
D_3'
D_3''
E_3'
E_3''
E_5'
E_5''
F_1'
F_1''
G_1'

Problem. Removing "accident" graphs from $\mathcal{M}_\Sigma$

Theorem (Hall) Except for K_5, a 3-connected graph is non-planar iff it contains $K_{3,3}$.

K_5 is an accident!

Objective. Find $\mathcal{B} \subseteq \mathcal{A}_3$ such that:

With finitely many exceptions, a 3-connected graph is non-projective iff it contains a graph in $\mathcal{B}$

Theorem. There are precisely two minimal sets $\mathcal{B}$:

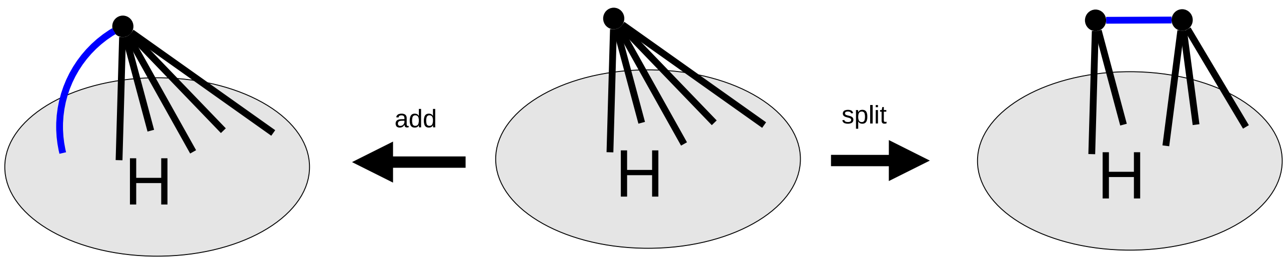
- $\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$ (21 exceptions)
- $\mathcal{A}_3 - \{ \quad B_7, \quad C_7, D_{17}\}$ (21 exceptions)

Proof. Using Splitter Theorem . . . .

Splitter Theorem. (Seymour) If

- G and H are 3-connected
- $K_4 \neq H < G \neq W_n$

then $G \geq H' \in \{H\text{-adds}, H\text{-splits}\}$.



Hall Theorem. If $G \neq K_5$ is 3-connected nonplanar then $G \geq K_{3,3}$.

Proof. Nonplanar $\Rightarrow$ $G \geq K_5$ or $K_{3,3}$

$\Rightarrow$ $G \geq K_5$

$\Rightarrow$ $G \geq K_5\text{-split} \geq K_{3,3}$. □

Theorem. There are precisely two minimal sets $\mathcal{B}$:

- $\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$ (21 exceptions)
- $\mathcal{A}_3 - \{\quad B_7, \quad C_7, D_{17}\}$ (21 exceptions)

Proof. Using Splitter Theorem . . . .

Objective. Find $\mathcal{B} \subseteq \mathcal{A}_3$ such that:

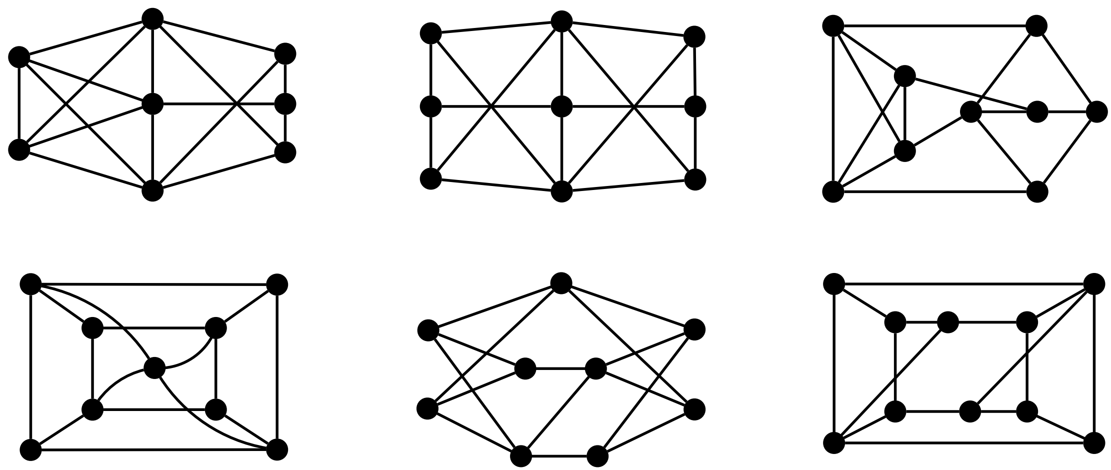
With finitely many exceptions, an internally 4-connected graph is non-projective iff it contains a graph in $\mathcal{B}$

Theorem (Our second main result). The following

$$\{D_3, E_5, E_{20}, E_{22}, F_1, F_4\}$$

is a minimum set $\mathcal{B}$.
(The largest exception has 14 vertices and 31 edges.)

<u>A different formulation</u>: An i-4-connected graph G with ≥ 15 vertices is projective iff G contains none of the following:



D_3, E_5, E_{20}, E_{22}, F_1, F_4

Proof.

Splitter Theorem.

If $G \geq H$, both i-4-c, and $|V(G)| > |V(H)|$,
then $G \geq H'$, where H' ......

Outer-Projective graphs.

A graph G is *outer-projective* if G admits a projective drawing such that there is a face incident with all vertices.

Observation. G is outer-projective iff $\hat{G}$ is projective.

Corollary. For outer-projective graphs,
the set $\mathcal{F}$ of forbidden minors consists of precisely
minimal graphs in $\{G - v : G \in \mathcal{A}, v \in V(G)\}$

Archdeacon, Hartsfield, Little, Mohar (1998): $|\mathcal{F}| = 32$

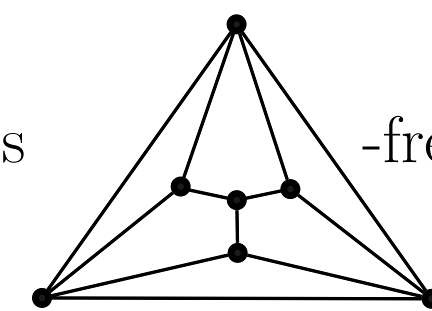
Theorem.

(1) A connected G is OP iff G is $\mathcal{F}_1$-free; $|\mathcal{F}_1| = 29$

(2) A 2-connected G is OP iff G is $\mathcal{F}_2$-free; $|\mathcal{F}_2| = 23$

(3) A 3-connected G is OP iff G is $\mathcal{F}_3^*$-free; $|\mathcal{F}_3^*| = 9$

(4) An i-4-connected G with $|G| \geq 9$ is OP iff

G is  -free.